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Asymptotic Expansions of Navier-Stokes Solutions in Three Dimensions for Large Distances

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PREFACE

The major portion of this research was carried out by the author at the Guggenheim Aeronautical Laboratory, California Institute of Technology, with the support of the Air Force Office of Scientific Research under Contract AF 49 (638)-521. It was completed at the Jet Propulsion Laboratory, National Aeronautics and Space Administration, under Contract NASw-6.

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ABSTRACT

15784 author

This Report studies the stationary flow field at large distances from a finite obstacle moving uniformly in a viscous, incompressible fluid. The principal results consist of asymptotic expansions, uniformly valid for large distances, of the velocity and the pressure of the flow field.

The expansion procedure employed is based upon the introduction of a small, extraneous parameter; the construction is thus recast as a perturbation analysis for small values of the parameter. Owing to the presence of a viscous wake, the perturbation is in general a singular one, and is treated accordingly, using methods developed for related hydrodynamical problems. It is found that the procedures needed for the three-dimensional case differ in no significant ways from the corresponding problem in two dimensions. However, the actual construction of the expansion is in general very different in the multidimensional problem, owing to the possible existence of strong crossflow in the wake region.

The calculated results include the following: for the case of axially symmetric flow, a uniformly valid expansion of the velocity to order r^{-2} inclusive and of the pressure to order r^{-3} inclusive, r being the distance from the obstacle; for the general case, an expansion of the velocity to order $r^{-3/2}$ inclusive and of the pressure to order r^{-2} inclusive. Author

I. INTRODUCTION

Problems related to expansions of Navier-Stokes solutions for large distances have been discussed by a number of authors, especially by Imai (Ref. 1) and by Chang (Ref. 2). In particular, it has been shown in Ref. 2 that the basic problem of construction may be treated by hydrodynamical expansion procedures of the type discussed and illustrated by Lagerstrom, Cole, and Kaplun (Ref. 3 and 4). In the present Report, the methods of Ref. 2, 3, and 4 are applied to axially symmetric and to strictly three-dimensional Navier-Stokes solutions. Expansion procedures for the three-dimensional case are discussed and several terms of the expansion are given. It is pointed out also that the procedures are slightly different for the case of three (or more) dimensions, owing to certain changes in the nature of crossflow and in the role of the pressure (see Section IV-A). The axially symmetric case, regarded as a problem in two dimensions, is discussed separately (Part III).

A certain class of Navier-Stokes solutions will be studied. The basic problem in mind is that of a stationary, viscous, incompressible flow past a finite three-dimensional solid, which tends to a uniform stream at large distances and satisfies the no-slip condition at the solid. Assuming that such a solution is given, we are interested in an asymptotic expansion of the solution for large distance at a fixed Reynolds number ($\mathfrak{Re}$), more precisely, in an asymptotic expansion valid to all orders r^{-n} as $r \to \infty$, where r is the distance from the origin. The problem studied here, however, will be of a slightly different nature. In the first place, the class of Navier-Stokes solutions studied will be somewhat larger: Given an asymptotic series, it is difficult to determine whether the related Navier-Stokes solutions contain a "solid," i.e., a closed streamsurface. On the other hand, a certain class of Navier-Stokes solutions is related to our series. In the second place, in the present Report we shall be concerned exclusively with the problem of construction of the series, which, of course, is only a part of the complete problem of asymptotic equality. The special nature of the relationship between our series and the class of Navier-Stokes solutions studied (be it, for example, that of actual asymptotic equality or even that of total equality) is then immaterial for our purposes. On the other hand, a statement of the intended validity of our results is desirable. The class of Navier-Stokes solutions studied and the sense in which our results are intended to be valid are described in Sections II-A and II-B.

The methods of Ref. 2 will be used: An extraneous nondimensional parameter ϵ (also called the "artificial parameter") is introduced into the exact solution in such a manner that the expansion for large r

may be replaced by a parameter-type expansion for small ϵ. In the present problem, ϵ may be regarded as the ratio of a characteristic length to the length of an extraneous standard of measurement. An "outer" and an "inner" expansion are then constructed, representing respectively the repeated applications of an "outer" and an "inner" limit process. The outer expansion is valid for large distances exclusive of the wake, while the inner expansion is valid in the wake. The regions of validity of the two expansions overlap in the sense of Ref. 4. A "composite" expansion, uniformly valid for large distances, may then be constructed from the two principal expansions (see Sections III-J and IV-I). An advantage of the parametric procedure is that one is first led to approximate partial differential equations, of considerable intuitive importance, while coordinate-type procedures would lead directly to ordinary differential equations.

A number of short cuts will be used in the course of the construction. However, the construction procedures are explained in Part II, where references are also given concerning points which require more elaborate discussion. In particular, reference will be made to two principles: (1) the principle of eliminability and (2) the principle of transcendental decay of vorticity. The two principles are discussed in Sections II-C and II-D.

II. THE EXACT SOLUTIONS AND THE EXPANSION PROCEDURES

A. The Exact Solutions

We consider stationary flows of a viscous, incompressible fluid in three dimensions. The following notation is used:

$$\mathbf{q} = \text{velocity}; \quad p = \text{pressure}; \quad x_i = \text{Cartesian coordinates},$$

$$x_1 = x, \quad x_2 = y, \quad x_3 = z, \quad r^2 = \sum_1^3 (x_i)^2; \tag{1}$$

$$\rho = \text{density} = \text{constant}; \quad \nu = \text{kinematic viscosity}.$$

The governing equations are the Navier-Stokes equations:

$$(\mathbf{q} \cdot \nabla)\,\mathbf{q} = -\frac{1}{\rho}\,\nabla p + \nu \nabla^2 \mathbf{q} \tag{2a}$$

$$\nabla \cdot \mathbf{q} = 0 \tag{2b}$$

Without loss of generality, we pass directly to the nondimensional form of the Navier-Stokes equations:

$$(\mathbf{q}^* \cdot \nabla^*)\,\mathbf{q}^* = -\nabla^* p^* + \frac{1}{\mathcal{R}_e}\,\nabla^{*2}\mathbf{q}^* \qquad (0 < \mathcal{R}_e < \infty) \tag{3a}$$

$$\nabla^* \cdot \mathbf{q}^* = 0 \tag{3b}$$

$$\nabla^* = \left(\frac{\partial}{\partial x_i^*}\right) \tag{3c}$$

The transformation

$$\mathbf{q} = U\,\mathbf{q}^* \tag{4a}$$

$$p = \rho U^2 p^* + P \tag{4b}$$

$$x_i = L x_i^* \tag{4c}$$

$$\mathcal{R}_e = \frac{UL}{\nu} \tag{4d}$$

sends every solution ($\mathbf{q}^*,p^*$) of Eq. (3) into a family of solutions of Eq. (2) which depends on the dimensional parameters U, L, P, ρ, ν; and, conversely, every solution of Eq. (2) may be so obtained. The question of the existence of a *characteristic* length for a given solution of Eq. (2) is thus expelled from our considerations. We shall consider solutions of Eq. (3) which satisfy the following conditions:

1. There exists a sphere S such that $\mathbf{q}^*$ and p^* are regular outside S and continuous at infinity.

2. At infinity,

$$\mathbf{q}^* = \mathbf{i}, p^* = 0 \tag{5a}$$

We shall also require

3.
$$\oint \mathbf{q}^* \cdot d\mathbf{s}^* = 0 \tag{5b}$$

Condition 3 is not essential, but leads to a number of well-known dynamical relations concerning flow at large distances. Above, solutions of Eq. (3) are regarded as distinct for distinct values of the Reynolds number $\mathcal{R}_e$. Hence a solution ($\mathbf{q}^*, p^*$) of Eq. (3) is a function of the x_i^* only. Expansions will be constructed for $r^* \to \infty$.

B. Limits and Expansions

Given a solution ($\mathbf{q}^*, p^*$) one may introduce an extraneous parameter ϵ and new independent variables $\tilde{x}_i$ or $\bar{x}_i$ by the substitutions

$$\tilde{x}_i = \epsilon x_i^* \qquad \text{(Outer variables)} \tag{6a}$$

$$\left.\begin{aligned} \tilde{x} &= \bar{x} \\ \tilde{y} &= \epsilon^{1/2}\bar{y} \\ \tilde{z} &= \epsilon^{1/2}\bar{z} \end{aligned}\right\} \qquad \text{(Inner variables)} \tag{6b}$$

The parameter ϵ and the variables $\tilde{x}_i$ admit the following evident interpretation: $\tilde{x}_i = x_i/R$ are the coordinates of a point referred to an extraneous standard of length measurement, of length R. In the outer limit process, R and $\tilde{x}_i$ are fixed while the characteristic length, $L = \epsilon R$, is decreased to zero; the Reynolds number, $\mathcal{R}e = UL\ \nu$, is held fixed in the process. By repeated applications of the outer and inner limit processes to a given flow quantity W, one obtains (provided the limits exist) two expansions, outer and inner, of the form

$$W \sim \sum_{i=0} \delta_i(\epsilon)\, \tilde{w}_i(\tilde{x}_i) \qquad \text{(Outer expansion)} \qquad (7a)$$

$$W \sim \sum_{i=0} \delta_i(\epsilon)\, \bar{w}_i(\bar{x}_i) \qquad \text{(Inner expansion)} \qquad (7b)$$

Here $\{\delta_i(\epsilon)\}$ $(i = 0, 1, 2, \cdots)$ is a sequence of functions (called orders or gauge functions) such that

$$\lim_{\epsilon \to 0} \frac{\delta_{i+1}}{\delta_i} = 0 \qquad (i = 0, 1, 2, \cdots) \qquad (7c)$$

If domains of validity of partial sums of expansions (7) overlap, as discussed in Ref. 4, it is then possible to construct a composite expansion which is uniformly valid for $r^* \to \infty$.

The terms of expansions (7) are defined by the form of the expansions (7a) and (7b), except for the trivial freedom allowed in the choice of δ's. The "form" of the expansion, understood in an extended sense to include the stipulated domains of uniform validity, also determines the equations and the boundary and matching conditions which the terms must satisfy. (The equations may be found by a formal substitution of the series in Eq. 3).

In the present Report, matching series of the form (7) will be constructed on the basis of equations, boundary, and matching conditions (and an additional condition, namely, that of eliminability of the extraneous parameter, see Section II-C). The existence of an actual asymptotic expansion of the form (7) is not absolutely essential and is in fact not stipulated in the present thesis; this question is discussed explicitly below. Different quantities, W, will be introduced as needed in each case (see Eq. 15, 16, 53, and 56, which give the explicit forms for the several cases treated). The gauge functions, $\delta_i(\epsilon)$, will be determined iteratively, but not in strictly consecutive order; the iteration process involves "switchback" (as does that of Chang, Ref. 2). The δ's will be reindexed in the form

$$\delta_i = \delta_{\nu_i} \tag{7d}$$

where the ν's are chosen as convenient in the iteration process (each ν represents what is regarded as a definite step in the procedure).

The stipulated domains of uniform validity of expansions (7a) and (7b) may be described as follows: under the outer limit process, $\mathbf{q}^*$ tends to $\mathbf{i}$ and p^* tends to zero uniformly over the entire $\tilde{x}$-space, excluding the point at the origin $\tilde{x}_i = 0$. This is evident by hypothesis (i.e., from the boundary conditions, Eq. 5a). However, in general, the outer expansion is not uniform at the positive $\tilde{x}$-axis. This is due to the presence of singular perturbations which represent the decay of the wake and are, in general, of order $\epsilon^{1/2}$ (e.g., in the presence of lift) or of order ϵ (drag but no lift). The inner expansion, on the other hand, should be valid in the wake region, or, more precisely, in the right half of the $\bar{x}_i$-space, excluding the plane $\bar{x} = 0$. The regions of validity of expansions (7a) and (7b) should overlap for large $\bar{\rho}$ (small $\tilde{\rho}$) (see Eq. 12 and 14). The non-uniformity of the inner expansion at the plane $\bar{x} = 0$ is not important; it is stipulated that the outer expansion is valid at that plane, excluding the point at the origin $\bar{x}_i = 0$. Hence, the two expansions, inner and outer, should match also for small $\bar{x} > 0$. An additional stipulation will be made in order to derive the boundary conditions at infinity for the outer expansion: it is stipulated that the outer expansion is uniform at infinity, excluding only the positive $\tilde{x}$-axis, and that the two expansions, outer and inner, jointly cover the point at infinity.

In this Report, matching series of the form (7) are constructed on the basis of the associated equations and conditions. The results are intended to be valid in the following sense:

1. For every partial sum of the expansion, and for every choice of the arbitrary constants of the series, there should exist a related Navier-Stokes solution of the class defined in Section II-A (i.e., an "exact solution").

2. Whenever an exact solution has an expansion of the form (7), then the expansion should be given correctly by our results.

Statements 1 and 2 tell us in what sense our series is "correct" or "grossly incorrect." No other questions enter the construction process. The guidance supplied by statements 1 and 2 is, however, needed in the construction procedure.

In the first place, statement 2 determines the associated equations and boundary conditions for each term of the series. This has already been discussed in a preceding paragraph (see Section II-B). In accordance with statement 2, therefore, we should admit, for each term of (7), the most general expression that is allowed by the boundary, matching, and eliminability conditions (provided only that the expression does not contradict statement 1). In the present expansions, there exist complementary solutions of the associated equations (called "eigensolutions") which satisfy homogeneous boundary and matching conditions and the condition of eliminability (see Section II-C). (Here, by "boundary" and "matching" conditions we understand, of course, those conditions which are *governing* for the term in question, i.e., those conditions which may be derived from the overlap principle or from condition (5a). Thus, for example, it is not required that an outer eigensolution vanish at the positive $\tilde{x}$-axis, since no such fact derives from the basic premises.) The most general expression for each term is obtained by finding all possible eigensolutions that are admitted by the governing conditions for the term in question. In particular, it appears that the question of so-called "intermediate orders" or "phantom terms" (i.e., the question of existence of terms of order not listed explicitly in each case) may, in principle, be decided entirely on the basis of the conditions. This is illustrated in Section III-D.

In accordance with statement 1, it is necessary to take into account the possible "integrated effects" of the forcing term: the domains of validity do not become evident until such effects are considered. In general, the "integrated effects" include the possibility of resonance and also other possibilities, e.g., that the solution may be rendered multivalued. In the present case, however, an estimate of the effects of the forcing term is provided at each step by a subsequent term of the series, and will be here, in principle, taken into account by inspection. The forcing terms appear to be entirely harmless within the stipulated domains of validity.

It is believed that no other essential considerations need to be taken into account: although the theory of constructions such as the present ones has never been fully discussed, it has been suggested to the author that a favorable estimate of the possible effects of a small, arbitrary forcing term is probably sufficient, i.e., may lead directly to a rigorous proof of statement 1.

C. The Principle of Eliminability

All governing conditions for our series are derived from the definition of the exact solutions, by means of the hypotheses on the validity of the series (cf. Section II-B). It should be noted, therefore, that the definition has changed: after the parameter has been introduced, we are dealing with functions $\mathbf{q}^*(\tilde{x}_i;\epsilon)$ and $p^*(\tilde{x}_i;\epsilon)$ which satisfy the Navier-Stokes equations (Eq. 3), the conditions 1, 2, and 3 (Section II-A, Eq. 5), and the following eliminability conditions:

$$\mathbf{q}^*(x_i^*\epsilon;\epsilon) = \lim_{\epsilon\to 0} \mathbf{q}^*(x_i^*\epsilon;\epsilon) \tag{8a}$$

$$p^*(x_i^*\epsilon;\epsilon) = \lim_{\epsilon\to 0} p^*(x_i^*\epsilon;\epsilon) \tag{8b}$$

Conditions (8) state that ϵ is eliminated by the substitution of $x_i^*\epsilon$ for $\tilde{x}_i$. Of a partial sum S_n of series (7a) and (7b) it is then required that, in its stipulated domain of validity, S_n be expressible in the form

$$S_n = f(x_i^*) + R(x_i^*;\epsilon) \tag{9a}$$

where

$$\frac{R}{\delta_n(\epsilon)} \text{ is uniformly small.} \tag{9b}$$

The condition (9) will be referred to as the eliminability principle; it is a governing condition for the series.

D. The Principle of Rapid (Transcendental) Decay of Vorticity

It is a certain (although possibly unproved) hydrodynamical fact that, for a finite or semi-infinite solid in a uniform stream, the vorticity decays at an exponential rate with distance outside the wake or the boundary layer, as the case may be. It is a corollary to this fact that

> The vorticity must also decay transcendentally in every term of our inner expansions, (7b), as $\bar{x} \to 0$. (10)

The term "principle of rapid decay" is in general applied to both the theorem and the corollary, and the corollary is also often understood to stipulate $\bar{\rho} \to \infty$ rather than $\bar{x} \to 0$. In this Report, by "principle of rapid decay" we shall understand the corollary (10) rather than the theorem, and $\bar{x} \to 0$ rather than $\bar{\rho} \to \infty$. The reason is the following: the corollary is a direct consequence of the matching conditions at the plane $\bar{x} = 0$, which supply the *initial conditions* for the partial differential equations involved. (Those solutions which decay algebraically have nonzero vorticity at the plane $\bar{x} = 0$ and, hence, cannot be matched to the outer solutions there.)

The "principle of rapid decay" is not a governing condition, but rather a consequence of the governing matching conditions at $\bar{x} > 0$. In the present construction it is almost equally convenient to use either corollary (10) or the proper governing conditions, since the equivalence of the two is evident at every step. However, reference to the corollary (10) makes it possible to reject algebraic solutions at sight.

III. THE ASYMPTOTIC EXPANSIONS FOR THE AXIALLY SYMMETRIC CASE

A. The Principal Expansions

In the present Part we shall consider solutions which are symmetric about the x^*-axis. The axially symmetric problem will be treated as a problem in two dimensions (i.e., in two independent variables). For discussion of the axially symmetric case as a "special case" of the general three-dimensional problem, see Part IV.

The velocity field $\mathbf{q}^*$ may be expressed in the form

$$\mathbf{q}^* = u^* \mathbf{i}_x + v^* \mathbf{i}_\rho + w^* \mathbf{i}_\theta \tag{11}$$

where $(\mathbf{i}_x, \mathbf{i}_\rho, \mathbf{i}_\theta)$ is a right-handed orthonormal set of vectors corresponding to cylindrical polar coordinates x^*, ρ^*, θ, where

$$\rho^* = (y^{*2} + z^{*2})^{1/2}, \qquad \theta = \tan^{-1}\left(-\frac{y^*}{z^*}\right), \qquad 0 \le \theta \le 2\pi \tag{12}$$

The governing equations for the axially symmetric case are obtained by passage to polar coordinates (see Eq. A-1, Appendix A) and by putting

$$\frac{\partial u^*}{\partial \theta} = \frac{\partial v^*}{\partial \theta} = \frac{\partial w^*}{\partial \theta} = \frac{\partial p^*}{\partial \theta} = 0 \tag{13}$$

in Eq. (A-1). The independent variables are then x^* and ρ^*. The corresponding inner and outer variables are

$$\tilde{x} = \epsilon x^*, \qquad \tilde{\rho} = \epsilon \rho^* \qquad \text{(Outer variables)} \tag{14a}$$

$$\bar{x} = \epsilon x^*, \qquad \bar{\rho} = \epsilon^{1/2} \rho^* \qquad \text{(Inner variables)} \tag{14b}$$

The limit process expansions to be obtained below are of the form:

Outer:

$$\mathbf{q}^* = \mathbf{i} + \epsilon^2 \mathbf{q}_2(\tilde{x}, \tilde{\rho}) + \epsilon^3 \log \epsilon \;\; \mathbf{q}_{3a}(\tilde{x}, \tilde{\rho}) + \epsilon^3 \mathbf{q}_3(\tilde{x}, \tilde{\rho}) + o(\epsilon^3) \tag{15a}$$

$$p^* = \epsilon^2 p_2(\tilde{x}, \tilde{\rho}) + \epsilon^3 \log \epsilon \;\; p_{3a}(\tilde{x}, \tilde{\rho}) + \epsilon^3 p_3(\tilde{x}, \tilde{\rho}) + o(\epsilon^3) \tag{15b}$$

Inner:

$$u^* = 1 + \epsilon u_1(\bar{x}, \bar{\rho}) + \epsilon^2 \log \epsilon \;\; u_{2a}(\bar{x}, \bar{\rho}) + \epsilon^2 u_2(\bar{x}, \bar{\rho}) + o(\epsilon^2) \tag{16a}$$

$$\bar{v} = \epsilon^{-1/2} v^* = \epsilon v_1(\bar{x}, \bar{\rho}) + \epsilon^2 \log \epsilon \;\; v_{2a}(\bar{x}, \bar{\rho}) + \cdots \tag{16b}$$

$$\bar{w} = \epsilon^{-1/2} w^* = \epsilon w_1(\bar{x}, \bar{\rho}) + \cdots \tag{16c}$$

$$p^* = \epsilon p_1(\bar{x}, \bar{\rho}) + \epsilon^2 p_2(\bar{x}, \bar{\rho}) + \epsilon^3 \log \epsilon \;\; p_{3a}(\bar{x}, \bar{\rho}) + \epsilon^3 p_3(\bar{x}, \bar{\rho}) + o(\epsilon^3) \tag{16d}$$

B. A Remark Concerning the Outer Expansion: Irrotationality of the Outer Flow Field

We shall show that the outer flow field is irrotational to all finite orders ϵ^n as $\epsilon \to 0$, or equivalently, to all finite orders r^{*-n} as $r^* \to \infty$.

The general Navier-Stokes equations in outer variables are

$$\mathbf{q}^* \cdot \tilde{\nabla} \mathbf{q}^* + \tilde{\nabla} p^* = \frac{\epsilon}{\mathcal{R}e} \tilde{\nabla}^2 \mathbf{q}^* \tag{17a}$$

$$\tilde{\nabla} \cdot \mathbf{q}^* = 0 \tag{17b}$$

$$\tilde{\nabla} = \left(\frac{\partial}{\partial \tilde{x}_i} \right) \tag{17c}$$

If the outer expansion (15) is inserted into Eq. (17), a term-by-term calculation may in principle be carried out. It is possible, however, to obtain the result by a direct argument. One first observes that an irrotational, solenoidal vector field $\mathbf{q}^*$ is a solution of the Navier-Stokes equations. In particular, the term of order ϵ in Eq. (17a) is zero for any such solution. We now impose the condition that the vorticity of the related Navier-Stokes solutions be zero at upstream infinity. This determines the outer limit to be an irrotational flow. In the succeeding calculations, the right-hand side of Eq. (17a) will always vanish. We conclude that the outer expansion of the vector velocity consists of a series of terms, each of which is the gradient of a harmonic function of $\tilde{x}_i$. The outer expansion of pressure is then a consequence of the constancy of total head in potential flows.

In particular, if the expansions (15a) and (15b) are inserted into Bernoulli's equation, one finds

$$\tilde{p}_2 = -\mathbf{q}_2 \cdot \mathbf{i}_x, \qquad p_{3a} = -\mathbf{q}_{3a} \cdot \mathbf{i}_x, \qquad p_3 = -\mathbf{q}_3 \cdot \mathbf{i}_x \tag{18}$$

C. Equations for the Inner Terms

The exact equations for the axially symmetric case, written in inner variables, are

$$H(u') + \frac{\partial p^*}{\partial \bar{x}} = -u' \frac{\partial u'}{\partial \bar{x}} - \bar{v} \frac{\partial u'}{\partial \bar{\rho}} + \frac{\epsilon}{\mathcal{R}e} \frac{\partial^2 u'}{\partial \bar{x}^2} \tag{19a}$$

$$\frac{\partial(\bar{\rho} u')}{\partial \bar{x}} + \frac{\partial(\bar{\rho}\,\bar{v})}{\partial \bar{\rho}} = 0 \tag{19b}$$

$$\frac{\partial p^*}{\partial \bar{\rho}} = -\epsilon \left[H_1(\bar{v}) + u' \frac{\partial \bar{v}}{\partial \bar{x}} + \bar{v} \frac{\partial \bar{v}}{\partial \bar{\rho}} - \frac{\bar{w}^2}{\bar{\rho}} \right] + \frac{\epsilon^2}{\mathcal{R}e} \frac{\partial^2 \bar{v}}{\partial \bar{x}^2} \tag{19c}$$

$$H_1(\bar{w}) = -u' \frac{\partial \bar{w}}{\partial \bar{x}} - \bar{v} \frac{\partial \bar{w}}{\partial \bar{\rho}} - \frac{\bar{v}\,\bar{w}}{\bar{\rho}} + \frac{\epsilon}{\mathcal{R}e} \frac{\partial^2 \bar{w}}{\partial \bar{x}^2} \tag{19d}$$

where u', $\bar{v}$, and $\bar{w}$ are defined by

$$u' = u^* - 1, \qquad \bar{v} = \epsilon^{-1/2} v^*, \qquad \bar{w} = \epsilon^{-1/2} w^* \tag{19e}$$

and the linear differential operators H and H_1 are defined by

$$H = \frac{\partial}{\partial \bar{x}} - \frac{1}{\mathcal{R}_e}\left(\frac{\partial^2}{\partial \bar{\rho}^2} + \frac{1}{\bar{\rho}}\frac{\partial}{\partial \bar{\rho}}\right) \tag{19f}$$

$$H_1 = H + \frac{1}{\mathcal{R}_e}\frac{1}{\bar{\rho}^2} \tag{19g}$$

Each term of Eq. (19a-d) tends to a uniform limit under the inner limit process. By a repeated application of the inner limit process to the exact governing equations, or equivalently, by formal substitution of the inner expansion, one obtains governing "approximate" equations satisfied by the terms of the inner expansion. The approximate partial differential equations are

$$H(u_\nu) + \frac{\partial p_\nu}{\partial \bar{x}} = \frac{1}{\mathcal{R}_e}\frac{\partial^2 u_{\nu-1}}{\partial \bar{x}^2} + f_\nu \tag{20a}$$

$$\frac{\partial(\bar{\rho}\, u_\nu)}{\partial \bar{x}} + \frac{\partial(\bar{\rho}\, v_\nu)}{\partial \bar{\rho}} = 0 \tag{20b}$$

$$\frac{\partial p_\nu}{\partial \bar{\rho}} = g_\nu \tag{20c}$$

$$H_1(w_\nu) = \frac{1}{\mathcal{R}_e}\frac{\partial^2 w_{\nu-1}}{\partial \bar{x}^2} + h_\nu \tag{20d}$$

Note that ν is not necessarily an integer (cf. Eq. 16). The forcing terms f_ν, g_ν, and h_ν vanish for $\nu < 2$. Equations (20a) through (20c) form a simultaneous system; Eq. (20d) may be solved independently.

D. The Leading Terms of the Inner and Outer Expansions

The equations for u_1, v_1, p_1, and w_1 are

$$H(u_1) + \frac{\partial p_1}{\partial \bar{x}} = 0 \tag{21a}$$

$$\frac{\partial(\bar{\rho} u_1)}{\partial \bar{x}} + \frac{\partial(\bar{\rho} v_1)}{\partial \bar{\rho}} = 0 \tag{21b}$$

$$\frac{\partial p_1}{\partial \bar{\rho}} = 0 \tag{21c}$$

$$H_1(w_1) = 0 \tag{21d}$$

The relevant solutions are

$$u_1 = -\frac{a\mathfrak{R}_e}{4\pi}\frac{e^{-\sigma}}{\bar{x}}, \qquad v_1 = -\frac{a}{2\pi}\frac{\sigma e^{-\sigma}}{\bar{\rho}\,\bar{x}} \tag{22a}$$

$$w_1 = -\frac{m\mathfrak{R}_e}{4\pi}\frac{\sigma e^{-\sigma}}{\bar{\rho}\,\bar{x}}, \qquad p_1 = 0 \tag{22b}$$

where

$$\sigma = \frac{\mathfrak{R}_e}{4}\frac{\bar{\rho}^2}{\bar{x}} = \frac{\mathfrak{R}_e}{4}\frac{\rho^{*2}}{x^*} \tag{22c}$$

and a and m are arbitrary constants. As a consequence of condition (5b), the constant a may be related to the dimensionless drag experienced by any closed streamsurface; the constant m may be related to the moment (see Section IV-K). The constant a also represents the strength of a "viscous sink" placed at the origin.

This may be seen by considering the streamfunction $\psi_1(\bar{x},\bar{\rho})$ for the terms u_1 and v_1:

$$\psi_1 = \frac{a}{2\pi} e^{-\sigma}; \qquad u_1 = \frac{1}{\bar{\rho}} \frac{\partial \psi_1}{\partial \bar{\rho}}; \qquad v_1 = -\frac{1}{\bar{\rho}} \frac{\partial \psi}{\partial \bar{x}} \tag{23a}$$

One sees that, for $\bar{x} > 0$,

$$2\pi \left[\psi_1(\bar{x},\infty) - \psi_1(\bar{x},0)\right] = -a \tag{23b}$$

To obtain Eq. (22), one first observes that the principle of eliminability requires that p_1 be a constant multiple of $\bar{x}^{-1}$. By matching with the outer expansion of Eq. (15b), it follows that $p_1 = 0$. Also, again by eliminability,

$$u_1 = \frac{1}{\bar{x}} f(\sigma) \tag{24}$$

We require that $f(\sigma)$ be regular at $\sigma = 0$ and vanish exponentially as $\sigma \to \infty$. Inserting Eq. (24) into Eq. (21a), there results a second-order ordinary differential equation for f. It is shown in Appendix A that f is determined by the conditions stated above to be a constant multiple of $e^{-\sigma}$. This determines u_1 to within a multiplicative constant. Then v_1 may be found by integrating the continuity equation; the constant of integration is zero, since v_1 is regular on the line $\bar{\rho} = 0$.

The calculation of w_1 is similar to that of u_1 and will be omitted.

The outer terms of order ϵ^2 appear as a consequence of the existence of a nonzero drag. The term $\mathbf{q}_2$ is determined by requiring that the mass flux through any closed surface containing the solid be zero. This condition has been stated above by Eq. (5b). In order to balance the mass inflow in the wake (cf. Eq. 23b) a term representing the flow due to a potential source must appear in the outer expansion of $\mathbf{q}^*$. Thus,

$$\mathbf{q}_2 = -\frac{a}{4\pi} \tilde{\nabla} \left(\frac{1}{\tilde{r}}\right), \qquad \tilde{r} = (\tilde{x}^2 + \tilde{\rho}^2)^{1/2} \tag{25}$$

One notes that, if condition (5b) were relaxed, the multiplicative constant in Eq. (25) would be arbitrary.

The possibility of terms of orders other than those exhibited explicitly in Eq. (15) and (16) will not be discussed in full detail. However, we shall eliminate terms of all orders $\epsilon^{\alpha}(0 \leq \alpha < 2, \alpha \neq 1)$. This is accomplished by referring to the principle of transcendental decay: First we shall consider the outer expansion. Our solutions must be regular except at zero and infinity and possibly along the positive $\tilde{x}$-axis. It is evident also that the outer solutions are regular on the positive $\tilde{x}$-axis (except possibly at infinity) unless they match with nontrivial solutions of the homogeneous system (Eq. 21). Hence it is sufficient to restrict attention to (1) the homogeneous system (Eq. 21) and (2) solutions of Laplace's equation which are regular everywhere except at the origin and at infinity. We next discover that the outer solutions are regular at infinity since there the condition $\mathbf{q}^* = \mathbf{i}$ must be satisfied (in virtue of a basic stipulated region of validity of the outer expansion). Hence the solutions of Laplace's equation are poles and proceed in integral powers of $\tilde{r}^{-1}$ or equivalently in integral powers of ϵ.

Turning next to the homogeneous system (Eq. 21), we shall denote inner terms of order ϵ^{α} by subscript α. One sees immediately that p_{α} is a multiple of $\bar{x}^{-\alpha}$. By matching with the outer expansion, we conclude $p_{\alpha} = 0\,(0 \leq \alpha < 2)$. Also, we have $u_{\alpha} = \bar{x}^{-\alpha} f_{\alpha}(\sigma)$. From the results of Appendix A one finds that if $p_{\alpha} = 0$ and if u_{α} satisfies the condition of transcendental decay, then necessarily $\alpha = 1$. An analogous statement holds for w_{α}.

E. The Inner Terms of Order ϵ^2

In this section the terms u_2, v_2, p_2, and w_2 will be given. The relevant equations are

$$H(u_2) + \frac{\partial p_2}{\partial \bar{x}} = \frac{1}{\mathcal{R}_e} \frac{\partial^2 u_1}{\partial \bar{x}^2} + f_2 \tag{26a}$$

$$\frac{\partial(\bar{\rho} u_2)}{\partial \bar{x}} + \frac{\partial(\bar{\rho} v_2)}{\partial \bar{\rho}} = 0 \tag{26b}$$

$$\frac{\partial p_2}{\partial \bar{\rho}} = 0 \tag{26c}$$

$$H_1(w_2) = \frac{1}{\mathcal{R}_e} \frac{\partial^2 w_1}{\partial \bar{x}^2} + h_2 \tag{26d}$$

where the forcing functions f_2 and h_2 are defined by

$$f_2 = -\left(u_1 \frac{\partial u_1}{\partial \bar{x}} + v_1 \frac{\partial u_1}{\partial \bar{\rho}}\right)$$

$$= \alpha^2 \frac{e^{-2\sigma}}{\bar{x}^3} \tag{27a}$$

$$h_2 = -\left(u_1 \frac{\partial w_1}{\partial \bar{x}} + v_1 \frac{\partial w_1}{\partial \bar{\rho}} + \frac{v_1 w_1}{\bar{\rho}}\right)$$

$$= \alpha \beta \frac{\sigma e^{-2\sigma}}{\bar{\rho} \bar{x}^3} \tag{27b}$$

The constants appearing in the last equations are not new and are given by

$$\alpha = \frac{a \mathfrak{Re}}{4\pi}, \qquad \beta = \frac{m \mathfrak{Re}}{4\pi} \tag{27c}$$

The correct solution of Eq. (26c) is

$$p_2 = c \bar{x}^{-2} \tag{28}$$

where c is a constant. Noting that the inner expansion of $\mathbf{i}_x \cdot \mathbf{q}_2$ is

$$\mathbf{i}_x \cdot \mathbf{q}_2 = \frac{a}{4\pi} \frac{1}{\bar{x}^2} + O(\epsilon) \tag{29}$$

one sees that, by matching of the series for the pressure,

$$c = -\frac{a}{4\pi} \tag{30}$$

Solving now for u_2, it is convenient to divide the forcing term f_2 into two parts:

$$f_2 = f_2^{(1)} + f_2^{(2)} \tag{31a}$$

where

$$f_2^{(1)} = \frac{\alpha^2}{\bar{x}^3} \left[e^{-2\sigma} + \frac{1}{4} (\sigma - 1) e^{-\sigma} \right] \tag{31b}$$

$$f_2^{(2)} = \frac{\alpha^2}{4\bar{x}^3} (1 - \sigma) e^{-\sigma} \tag{31c}$$

If $f_2^{(1)}$ appears in place of f_2 in Eq. (26a), there exists a particular solution from which the parameter is strictly eliminable. It is

$$u_2^{(1)} = \frac{a}{4\pi} \frac{1}{\bar{x}^2} + \frac{\bar{x}}{\mathcal{R}_e} \frac{\partial^2 u_1}{\partial \bar{x}^2} + \frac{\alpha^2}{4} \frac{1}{\bar{x}^2} \{(1 - \sigma) e^{-\sigma} [\mathrm{Ei}(-\sigma) - \log \sigma] - 2 e^{-\sigma} - e^{-2\sigma}\} \tag{32}$$

The notation for the exponential integral is that of Ref. 5. A particular integral for the forcing term $f_2^{(2)}$ is

$$\frac{\alpha^2}{4} \frac{\log \bar{x}}{\bar{x}^2} (1 - \sigma) e^{-\sigma} \tag{33}$$

as is easily seen from the fact that

$$\frac{1}{\bar{x}^2} (1 - \sigma) e^{-\sigma} \tag{34}$$

is an eigensolution. Using this result, we arrive at the general solution of Eq. (26a):

$$u_2 = u_2^{(1)} + u_2^{(2)} \tag{35a}$$

$$u_2^{(2)} = \frac{\alpha^2}{4} \frac{\log \bar{x}}{\bar{x}^2} (1 - \sigma) e^{-\sigma} + \frac{a_1}{\bar{x}^2} (1 - \sigma) e^{-\sigma} \tag{35b}$$

where $u_2^{(1)}$ is given by Eq. (32) and a_1 is an arbitrary constant. However, owing to the presence of the term involving $\log \bar{x}$ in Eq. (35b), ϵ is not eliminable from $\epsilon^2 u_2$. This will necessitate the introduction of a term of intermediate order (see Section III-F).

Integrating the continuity equation and adjusting the constant of integration so as to make v_2 regular at $\bar{\rho} = 0$, one finds

$$v_2 = v_2^{(1)} + v_2^{(2)} \tag{36a}$$

where

$$v_2^{(1)} = \frac{a}{4\pi} \frac{\bar{\rho}}{\bar{x}^3} + \frac{\bar{x}}{\mathcal{R}_e} \frac{\partial^2 v_1}{\partial \bar{x}^2} + \frac{\alpha^2}{2\mathcal{R}_e} \frac{\sigma}{\bar{\rho}\,\bar{x}^2} \{(2-\sigma)\,e^{-\sigma}\,[\mathrm{Ei}(-\sigma) - \log\sigma]$$

$$+ \frac{1}{\sigma}(e^{-\sigma} + e^{-2\sigma} - 2) - 3\,e^{-\sigma} - e^{-2\sigma}\} \tag{36b}$$

$$v_2^{(2)} = \frac{2a_1}{\mathcal{R}_e} \frac{\sigma}{\bar{\rho}\,\bar{x}^2} (2-\sigma)\,e^{-\sigma} + \frac{\alpha^2}{2\mathcal{R}_e} \frac{\log \bar{x}}{\bar{\rho}\,\bar{x}^2} \sigma(2-\sigma)\,e^{-\sigma} \tag{36c}$$

In order to determine w_2, one first notes that if u_p is defined by

$$u_p = \frac{\beta}{\alpha} u_2^{(1)} + \frac{\alpha\beta}{4} \frac{\log \bar{x}}{\bar{x}^2} (1-\sigma)\,e^{-\sigma} \tag{37a}$$

then

$$\frac{\partial}{\partial \bar{\rho}} H(u_p) = H_1\left(\frac{\partial u_p}{\partial \bar{\rho}}\right) = \frac{\beta}{\alpha} \frac{\partial f_2}{\partial \bar{\rho}} = -4h_2 \tag{37b}$$

Thus

$$w_2 = -\frac{1}{4} \frac{\partial u_p}{\partial \bar{\rho}} + \frac{\bar{x}}{\mathcal{R}_e} \frac{\partial^2 w_1}{\partial \bar{x}^2} + \frac{m_1}{\bar{\rho}\,\bar{x}^2} \sigma(2-\sigma)\,e^{-\sigma} \tag{38}$$

where m_1 is an arbitrary constant. Note that w_2 decays exponentially as $\bar{x} \to 0$. It is evident also that every partial sum of the inner expansion of $\bar{w}$ decays exponentially, since otherwise the pressure would be multi-valued.

F. Switchback: Terms of Intermediate Order

We have noted above that the parameter is not strictly eliminable from the terms $\epsilon^2 u_2$, $\epsilon^{5/2} v_2$, and $\epsilon^{5/2} w_2$. For example,

$$\epsilon^2 u_2(\bar{x},\bar{\rho}) = u_2(x^*,\rho^*) + \log \epsilon \cdot \frac{\alpha^2}{4} \, \frac{1}{x^{*2}} \, (1-\sigma)\, e^{-\sigma} \tag{39}$$

On the other hand, ϵ is eliminable from

$$\epsilon^2 u_2(\bar{x},\bar{\rho}) - \epsilon^2 \log \epsilon \cdot \frac{\alpha^2}{4} \, \frac{1}{\bar{x}^2} \, (1-\sigma)\, e^{-\sigma} \tag{40}$$

The second term appearing in Eq. (40), of order $\epsilon^2 \log \epsilon$, is precisely the term u_{2a} occurring in Eq. (16a). Terms which arise in this manner will be referred to as "switchback terms." Other examples of switchback have been discussed by Chang (Ref. 2). In our construction, switchback terms are uniquely determined by the principle of eliminability.

No attempt will be made to explain precisely the reason for switchback. It should be noted, however, that it is here a nonlinear phenomenon. Switchback terms do not appear in the expansions of solutions of the Oseen equations. Also, one observes that a particular integral for a forcing term which is an eigensolution will always require a switchback term (cf. Eq. 31c; see also Ref. 2, Appendix A).

By writing relations analogous to Eq. (39) for v_2 and w_2, the switchback terms v_{2a} and w_{2a} may be found.[1] In each case the switchback term is observed to be a solution of the homogeneous equation. This can be checked by deriving the equations for the inner terms of order $\epsilon^2 \log \epsilon$ in the usual way.

[1] The procedure just described appears to be sufficient to deal with all cases of switchback occurring in this Report. Since the necessary switchback terms will be obvious by inspection, explicit expressions are omitted here and in what follows.

G. The Outer Term of Order ϵ^3

One finds from Eq. (36) that the outer expansion of the inner expansion of v^* is

$$\tilde{v} = \epsilon^2 \frac{a}{4\pi} \frac{\tilde{\rho}}{\tilde{x}^3} - \epsilon^3 \frac{\alpha^2}{\mathcal{R}_e} \frac{1}{\bar{\rho}\,\bar{x}^2} + o(\epsilon^3) \tag{41}$$

The first term represents the flow in the wake, owing to the potential source, and matches with $\mathbf{q}_2$; the second matches with a term of order ϵ^3 in the outer expansion of $\mathbf{q}^*$. This term is given by

$$\mathbf{q}_3 = \tilde{\nabla}\phi_3, \qquad \tilde{\nabla}^2 \phi_3 = 0 \tag{42a}$$

where

$$\phi_3 = -\frac{\alpha^2}{2\mathcal{R}_e} \frac{1}{\tilde{r}^3} \left[\tilde{x} \log(\tilde{r} - \tilde{x}) + \tilde{r} - 2\tilde{x} \log \tilde{r}\right] + \frac{\tilde{a}_1}{2} \frac{\tilde{x}}{\tilde{r}^3} \tag{42b}$$

The first term on the right of Eq. (42b) is required by matching, and is associated with the switchback term $\mathbf{q}_{3a}$. The second is a potential dipole of arbitrary strength; this dipole is the eigensolution of order ϵ^3, i.e., it is the most general harmonic function homogeneous of degree 2 in the variables $\tilde{x}_i$ and regular everywhere except at the origin.

H. The Inner Term p_3

The term of order ϵ^3 in the inner expansion of pressure satisfies

$$\frac{1}{\bar{\rho}} \frac{\partial}{\partial \bar{\rho}} \left(\bar{\rho} \frac{\partial p_3}{\partial \bar{\rho}}\right) = -\frac{\partial^2 p_2}{\partial \bar{x}^2} + \frac{\partial f_2}{\partial \bar{x}} + \frac{1}{\bar{\rho}} \frac{\partial}{\partial \bar{\rho}} (\bar{\rho} f_2') \tag{43}$$

where f_2' is given by Eq. (27a) and where

$$f_2' = -u_1 \frac{\partial v_1}{\partial \bar{x}} - v_1 \frac{\partial v_1}{\partial \bar{\rho}} + \frac{w_1^2}{\bar{\rho}} \tag{44a}$$

$$= \left(\frac{3\alpha^2}{\Re_e} + \frac{\Re_e \beta^2}{4} \right) \frac{\sigma}{\bar{\rho}\, \bar{x}^3} e^{-2\sigma} \tag{44b}$$

One may obtain Eq. (43) by taking the divergence of Eq. (3a), expressing the resulting equation in inner variables, and expanding (cf. Eq. A-2). The solution is

$$p_3 = \frac{3a}{8\pi} \frac{\bar{\rho}^2}{\bar{x}^4} + \frac{\alpha^2}{2\Re_e} \frac{1}{\bar{x}^3} \left[2\,\mathrm{Ei}(-2\sigma) - \frac{1}{2} e^{-2\sigma} - 2 \log \sigma \right]$$

$$- \frac{\Re_e}{16} \beta^2 \frac{e^{-2\sigma}}{\bar{x}^3} + c_1 \frac{\log \bar{x}}{\bar{x}^3} + \frac{c_2}{\bar{x}^3} \tag{45}$$

The constants c_1 and c_2 can be determined by matching. One finds by expanding in the overlap domain that the inner and outer expansions of p^* match to order ϵ^3 inclusive if and only if

$$c_1 = \frac{2\alpha^2}{\Re_e} \tag{46a}$$

$$c_2 = \frac{\alpha^2}{2\Re_e} (2 \log \Re_e - 2 \log 2 - 5) \tag{46b}$$

where $\tilde{a}_1$ is the arbitrary constant appearing in Eq. (42b). Note that Eq. (45) is associated with a switchback term p_{3a}.

The term p_3 leads to the following conclusions regarding the inner expansion of p^* for the axially symmetric case: (1) to order ϵ^3 exclusive, the pressure penetrates the wake, i.e., all terms are functions of

$\bar{x}$ alone, and (2) owing to the nonlinear effect, there exists a term of order ϵ^3 which is discontinuous across the wake; the pressure discontinuity of this order is

$$p_3'(\bar{x},\infty) - p_3'(\bar{x},0) = \frac{1}{\bar{x}^3}\left[\frac{\mathfrak{R}_e\beta^2}{16} - \frac{\alpha^2}{4\mathfrak{R}_e}(4\gamma + 4\log 2 - 1)\right] \tag{47a}$$

where

$$p_3' = p_3 - \frac{3a}{8\pi}\frac{\bar{\rho}^2}{\bar{x}^4} \tag{47b}$$

and γ = Euler's constant.

I. Higher-Order Terms

Terms of higher order have not been studied in detail. However, it appears that the construction proceeds indefinitely, involving no change of the basic form (Eq. 15, 16) of the expansions. A suitable sequence of orders, $\{\delta_k(\epsilon)\}$, seems to consist of functions $\epsilon^i(\log\epsilon)^j$, where $i = 0, 1, 2, \cdots$, and $j = 0, 1, \cdots, i-1$. It has been brought to our attention that a similar conjecture seemed justified in Ref. 2.

Partial sums of the inner and outer expansions involve a number of arbitrary constants (e.g., a, m, a_1, $\tilde{a}_1$, m_1). A general fact may be pointed out concerning these constants: the number of arbitrary constants and the eigensolutions are unchanged if the Navier-Stokes equations are replaced by the Oseen equations. Hence, insofar as our construction procedure indicates, given any Navier-Stokes solution which has an expansion of the form considered here, one can find an Oseen solution with the same constants, and conversely.

J. The Composite Expansion

On the basis of the inner and outer expansions of the velocity and pressure, a *composite expansion* uniformly valid as $r \to \infty$ may be constructed (see Ref. 2, 3, and 4). The procedure used is essentially that of Ref. 2, and we state here only the results. Let $\tilde{E}_n(w)$ and $\bar{E}_n(w)$ denote partial sums of the inner and outer expansions (Eq. 15 and 16) which include terms of order ϵ^n. By virtue of Eq. (9), we then define

$$\tilde{E}_3(\mathbf{q}^*) = \mathbf{q}_o^*(x^*, \rho^*) \tag{48a}$$

$$\tilde{E}_3(p^*) = p_o^*(x^*, \rho^*) \tag{48b}$$

$$\epsilon^{1/2}\,\overline{E}_2(\overline{v}) = v_i^*(x^*, \rho^*) \tag{49a}$$

$$\epsilon^{1/2}\,\overline{E}_2(\overline{w}) = w_i^*(x^*, \rho^*) \tag{49b}$$

$$\overline{E}_2(u^*) = u_i^*(x^*, \rho^*) \tag{49c}$$

$$\overline{E}_3(p^*) = p_i^*(x^*, \rho^*) \tag{49d}$$

and

$$\mathbf{q}_i = u_i^*\,\mathbf{i}_x + v_i^*\,\mathbf{i}_\rho + w_i^*\,\mathbf{i}_\theta \tag{49e}$$

We also define the common parts

$$u_c^* = 1 + \frac{a}{4\pi}\,\frac{1}{x^{*2}} \tag{50a}$$

$$v_c^* = \frac{a}{4\pi}\,\frac{\rho^*}{x^{*3}} - \frac{\alpha^2}{\mathfrak{R}_e}\,\frac{1}{\rho^*\,x^{*2}} \tag{50b}$$

$$w_c^* = 0 \tag{50c}$$

$$p_c^* = -\frac{a}{4\pi}\,\frac{1}{x^{*2}} + \frac{3a}{4\pi}\,\frac{\rho^{*2}}{x^{*4}} - \frac{2\alpha^2}{\mathfrak{R}_e}\,\frac{\log\rho^*}{x^{*3}} + \frac{3\alpha^2}{\mathfrak{R}_e}\,\frac{\log x^*}{x^{*3}} + \frac{\alpha^2}{2\mathfrak{R}_e}\,(2\log 2 - 5)\,\frac{1}{x^{*3}} + \frac{\tilde{a}_1}{x^{*3}} \tag{50d}$$

and

$$\mathbf{q}_c^* = u_c^*\,\mathbf{i}_x + v_c^*\,\mathbf{i}_\rho + w_c^*\,\mathbf{i}_\theta \tag{51}$$

Then the functions

$$\mathbf{q}^*_{\text{comp}} = \mathbf{q}^*_o + (\mathbf{q}^*_i - \mathbf{q}^*_c) \tag{52a}$$

$$p^*_{\text{comp}} = p^*_o + (p^*_i - p^*_c) \tag{52b}$$

provide uniformly valid approximations (cf. Appendix B, Section I). The error may be estimated to be $O(r^{-2})$ in u^*, $O(r^{-5/2})$ in v^*, and $O(r^{-3})$ in p^*, uniformly as $r \to \infty$.

IV. THE ASYMPTOTIC EXPANSIONS FOR THE GENERAL THREE-DIMENSIONAL CASE

A. Crossflow and Pressure: The Multidimensional Problem

In this Part, expansion procedures are given and expansions are constructed for the general class of three-dimensional Navier-Stokes solutions defined in Section II-A. Axial symmetry is no longer required. However, a different expansion procedure suggests itself (and will be here adopted), owing to certain differences in the nature of crossflow and pressure in the wake region.

For problems involving a two-dimensional continuity equation (e.g., Ref. 2 or Part III), whenever u^* is given, the order of the crossflow (and, indeed, the crossflow itself) is fully determined by the continuity equation alone, together with a suitable boundary or matching condition. Consider, for example, the axially symmetric case: here $u^* - 1$ is of order ϵ in the wake region. Moreover, $\mathbf{q}^* - \mathbf{i}$ is of order ϵ^2 in the outer region. Hence, by the continuity equation and matching, v^* is of order $\epsilon^{3/2}$ in the wake. On the other hand, v^* is "small" with respect to the momentum equation, which then degenerates to $\partial p^*/\partial \bar{\rho} = 0$.

For a problem involving a multidimensional continuity equation, however, the situation is different. Here, the continuity equation does not determine the crossflow. Hence, the crossflow may be large with respect to the continuity equation and also with respect to the matching conditions for large $\bar{\rho}$. A striking illustration is given by the so-called "lifting case" (see Sections IV-D and IV-L). For the lifting case, the axial velocity disturbance, $u^* - 1$, remains of order ϵ (equivalently, r^{*-1}) in the wake region. The crossflow components, v^* and w^*, are, however, also of order ϵ and, hence, are obviously "large" with respect to the continuity equation. The large crossflow in the lifting case arises from the "horseshoe vortex" term of the wake expansion, $\mathbf{q}_1^+$ (Eq. 67b), which, in particular, vanishes for large $\bar{\rho}$ and therefore satisfies homogeneous matching conditions. On the other hand, v^* and w^* are no longer "small" with respect to the *momentum* equation; the governing system of approximate equations is a system of three simultaneous partial differential equations for v^*, w^*, and p^*, namely, the two momentum equations (58a and 58b) and the continuity equation (58c) in the crossflow plane. Hence, procedures for the multidimensional case involve a different sequence of steps, a sequence which, in a sense, is opposite to that of Part III and Ref. 2.

The three-dimensional procedures are introduced in Sections B and C below, by a statement of form of the expansion (Eq. 53 and 56), and are explained further in subsequent sections. The methods of Part III are certainly not a "special case" of the multidimensional methods; solutions with axial symmetry may, of course, be treated also by the three-dimensional method, and the results must agree numerically. However, the construction is entirely different. This engenders no paradox, since in Part III axial symmetry is introduced into the governing equations, while in the present Part it is a consequence of the boundary conditions.

B. The Inner and Outer Expansions

The three-dimensional solutions of Eq. (3), Part II, will be studied. Given the solution of Eq. (3), the x^*-axis is parallel to the velocity at infinity (see condition 5a). The y-axis is now chosen to be parallel to the component of the total force acting on the body (or upon a closed streamsurface in the flow field) which is normal to the x-axis; this normal component will be called the *lift*. The dimensionless variables x_i^*, $\tilde{x}_i$, and $\bar{x}_i$ are defined in Part II. Under the outer limit process ($\tilde{x}_i$ fixed), $\mathbf{q}^* \to \mathbf{i}$ uniformly. An expansion of the form

$$\mathbf{q}^* = \mathbf{i} + \epsilon^2 \mathbf{q}_2(\tilde{x}_i) + \epsilon^3 \log \epsilon \;\; \mathbf{q}_{3a}(\tilde{x}_i) + \epsilon^3 \mathbf{q}_3(\tilde{x}_i) + o(\epsilon^3) \tag{53a}$$

$$p^* = \epsilon^2 p_2(\tilde{x}_i) + \epsilon^3 \log \epsilon \;\; p_{3a}(\tilde{x}_i) + \epsilon^3 p_3(\tilde{x}_i) + o(\epsilon^3) \tag{53b}$$

will be constructed. The outer expansion is in general nonuniform in terms of order ϵ or higher at the positive $\tilde{x}$-axis. The nonuniformity represents the wake. The inner expansion will be treated differently here than in the axially symmetric case. The velocity $\mathbf{q}^*$ will be expressed as the sum of a crossflow velocity $\mathbf{q}^+$ and an axial velocity $u\,\mathbf{i}$:

$$\mathbf{q}^* = u\,\mathbf{i} + \mathbf{q}^+ \tag{54}$$

In order to obtain an indexing of terms such that each value of the index corresponds to a definite "step" in the construction, we introduce new dependent variables:

$$\bar{u} = \epsilon^{-1/2}(u^* - 1), \qquad \bar{p} = \epsilon^{-1/2} p^* \tag{55}$$

For $\bar{x}_i$ fixed, an expansion of the form

$$\bar{u} = \epsilon^{1/2} u_{1/2}(\bar{x}_i) + \epsilon \log \epsilon \; u_{1a}(\bar{x}_i) + \epsilon \, u_1(\bar{x}_i) + o(\epsilon) \tag{56a}$$

$$\mathbf{q}^+ = \epsilon \mathbf{q}_1^+(\bar{x}_i) + \epsilon^{3/2} \log \epsilon \; \mathbf{q}_{3/2a}^+(\bar{x}_i) + \epsilon^{3/2} \mathbf{q}_{3/2}^+(\bar{x}_i) + o(\epsilon^{3/2}) \tag{56b}$$

$$\bar{p} = \epsilon^{3/2} p_{3/2}(\bar{x}_i) + o(\epsilon^{3/2}) \tag{56c}$$

will be constructed. In particular, the plane of the variables $\bar{y}$ and $\bar{z}$ will be referred to as the "crossflow plane"; $\bar{x}$ may be regarded as a parameter in the inner expansion.

As explained in Part II, the terms of the outer expansion satisfy the Laplace equation,

$$\tilde{\nabla} \times \mathbf{q}_\nu = \tilde{\nabla} \cdot \mathbf{q}_\nu = 0 \tag{57}$$

The terms of the inner expansion satisfy the following equations:

$$\left(\frac{\partial}{\partial \bar{x}} - \frac{1}{\mathcal{R}e} \bar{\nabla}_+^2 \right) u_\nu + \frac{\partial p_\nu}{\partial \bar{x}} = f_\nu \tag{58a}$$

$$\left(\frac{\partial}{\partial \bar{x}} - \frac{1}{\mathcal{R}e} \bar{\nabla}_+^2 \right) \mathbf{q}_\nu^+ + \bar{\nabla}_+ p_\nu = g_\nu \tag{58b}$$

$$\bar{\nabla}_+ \cdot \mathbf{q}_\nu^+ = h_\nu \tag{58c}$$

where $\bar{\nabla}_+$ denotes the operator

$$\bar{\nabla}_+ = \left(\frac{\partial}{\partial \bar{y}} , \frac{\partial}{\partial \bar{z}} \right) \tag{59}$$

in the crossflow plane. The forcing terms f_ν, g_ν, and h_ν ($\nu > 0$) may be determined reiteratively from the exact equations (A-1). In particular, f_ν vanishes for $\nu < 1$, and g_ν and h_ν vanish for $\nu < 3/2$. Equations (58b) and (58c) constitute a simultaneous system for the crossflow and the pressure. At each step, the crossflow and the pressure may be determined first, and then the axial velocity may be found from Eq. (58a).

C. Fourier Analysis of the Inner Terms: Eigensolutions

It will be convenient to carry out the solution of Eq. (58) in cylindrical polar coordinates (see Section III-A). In particular, each term of the inner expansion may then be expressed as a terminating trigonometric series in θ, the coefficients of the series depending only upon $\bar{x}$ and $\bar{\rho}$. We define the crossflow terms v_ν and w_ν by

$$\mathbf{q}_\nu^+ = v_\nu \mathbf{i}_\rho + w_\nu \mathbf{i}_\theta \tag{60}$$

If $F_\nu(\bar{x}, \bar{\rho}, \theta)$ denotes any of the terms u_ν, v_ν, w_ν, or p_ν, expressed as a function of the variables $\bar{x}$, $\bar{\rho}$, and θ, we shall assume that there exists a Fourier expansion of F_ν of the form

$$F_\nu(\bar{x}, \bar{\rho}, \theta) = F_\nu^0(\bar{x}, \bar{\rho}) + \sum_{n=1}^{N} [F_\nu^n(\bar{x}, \bar{\rho}) \sin n\theta + \underline{F}_\nu^n(\bar{x}, \bar{\rho}) \cos n\theta] \tag{61}$$

where N is a suitable upper bound, depending upon ν (Section IV-E). The terms F_ν^n and $\underline{F}_\nu^n$ will be referred to simply as the orthogonal Fourier coefficients of order ν and degree n. The Fourier expansion (61) will be constructed for several of the inner terms (Sections IV-D through IV-H).

The Fourier coefficients u_ν^n, v_ν^n, $\underline{w}_\nu^n$, and p_ν^n ($n = 1, 2, \cdots$) satisfy the following system of equations:

$$H_n(u_\nu^n) + \frac{\partial p_\nu^n}{\partial \bar{x}} = f_\nu^n \tag{62a}$$

$$H_n(\bar{\rho}\, v_\nu^n) + \bar{\rho}\frac{\partial p_\nu^n}{\partial \bar{\rho}} = g_\nu^n \tag{62b}$$

$$\frac{\partial(\bar{\rho}\, v_\nu^n)}{\partial \bar{\rho}} \mp n\, \underline{w}_\nu^n = h_\nu^n \tag{62c}$$

$$L_n\,(p_\nu^n) = k_\nu^n \tag{62d}$$

where the differential operators H_n and L_n are defined by

$$H_n = \frac{\partial}{\partial \bar{x}} - \frac{1}{\Re} L_n \tag{62e}$$

$$L_n = \frac{\partial^2}{\partial \bar{\rho}^2} + \frac{1}{\bar{\rho}} \frac{\partial}{\partial \bar{\rho}} - \frac{n^2}{\bar{\rho}^2} \tag{62f}$$

The lower sign in Eq. (62c) is understood to apply to the equations for the orthogonal coefficients, obtainable from Eq. (62) by replacing u_ν^n by $\underline{u}_\nu^n$, $\underline{w}_\nu^n$ by w_ν^n, f_ν^n by $\underline{f}_\nu^n$, etc. The forcing terms f_ν^n, g_ν^n, h_ν^n, $\underline{f}_\nu^n$, etc., ($n = 1, 2, \cdots$) may be computed from the Fourier expansions of the forcing terms in Eq. (58).

The case $n = 0$ must be treated somewhat differently, since Eq. (62) is then a redundant system of equations. Equation (62b) is in this case replaced by the equation for w_ν^0:

$$H_1(w_\nu^0) = g_\nu^0 \tag{63}$$

Equations (62a), (62c), and (62d) may be solved for u_ν^0, v_ν^0, and p_ν^0, and w_ν^0 is given separately by Eq. (63).

The eigensolutions of the "inner equations" are defined to be the relevant solutions of the homogeneous system of equations corresponding to Eq. (58), that is, homogeneous solutions satisfying the principles of eliminability and transcendental decay of vorticity. If the eigensolutions are expanded in Fourier series of the form (61), there exists, as a consequence of these principles, a condition on the order and degree of the coefficients: an eigensolution which is of order $\delta_\nu(\epsilon) = \epsilon^\nu$ in the inner expansion (Eq. 56) will have a nonzero Fourier coefficient of degree n ($n = 0, 1, 2, \cdots$) if and only if

$$\nu - \frac{n}{2} - \frac{1}{2} = \text{non-negative integer.} \tag{64}$$

This result follows from the discussion in Section II of Appendix A. Whenever condition (64) is satisfied, the Fourier coefficients of the eigensolutions (denoted below by subscript e) are defined by

$$u^n_{\nu_e} = a^n_\nu W^n_{\nu+1/2} - p^n_{\nu_e} \tag{65a}$$

$$v^n_{\nu_e} = \frac{b^n_\nu}{\bar{\rho}} W^n_{\nu-1/2} + \frac{\bar{x}}{\bar{\rho}} p^n_{\nu_e} \tag{65b}$$

$$\underline{w}^n_{\nu_e} = \pm \frac{1}{n} \frac{\partial}{\partial \bar{\rho}} (\bar{\rho}\, v^n_{\nu_e}) \tag{65c}$$

$$p^n_{\nu_e} = \tilde{a}^n_\nu \bar{\rho}^n \bar{x}^{-(\nu+n/2+1/2)} \tag{65d}$$

whenever n = positive integer, and by

$$u^0_{\nu_e} = a^0_\nu W^0_{\nu+1/2} - p^0_{\nu_e} \tag{66a}$$

$$v^0_{\nu_e} = 0 \tag{66b}$$

$$w^0_{\nu_e} = b^0_\nu W^1_\nu \tag{66c}$$

$$p^0_{\nu_e} = \tilde{a}^0_\nu \bar{x}^{-(\nu+1/2)} \tag{66d}$$

for the case $n = 0$. The eigensolutions W^n_ν are defined in Appendix A (cf. Eq. A-11). The quantities a^n_ν, b^n_ν, and $\tilde{a}^n_\nu$ ($n = 0, 1, 2, \cdots$) are arbitrary constants; the orthogonal coefficients involve the constants $\underline{a}^n_\nu$, $\underline{b}^n_\nu$, and $\underline{\tilde{a}}^n_\nu$.

D. The Leading Terms Due to Lift and Drag

The leading terms of the inner expansion are (67a)

$$u_{1/2} = u^0_{1/2}(\bar{x},\bar{\rho})$$

(67b)

$$\mathbf{q}^+_1 = v^1_1(\bar{x},\bar{\rho}) \sin\theta \, \mathbf{i}_\rho + w^1_1(\bar{x},\bar{\rho}) \cos\theta \, \mathbf{i}_\theta$$

where

$$u^0_{1/2} = -\frac{a\mathfrak{R}_e}{4\pi}\frac{e^{-\sigma}}{\bar{x}} \tag{68a}$$

and

$$v^1_1 = \frac{b}{2\pi}\frac{1}{\bar{\rho}^2}(e^{-\sigma} - 1), \qquad w^1_1 = -\frac{b}{2\pi}\frac{1}{\bar{\rho}^2}[(2\sigma + 1)e^{-\sigma} - 1] \tag{68b}$$

Here a and b are arbitrary constants, related to the drag and lift, respectively (cf. Eq. 22a; see also Section IV-K).

To obtain Eq. (67) and (68), one first notes that the leading terms are eigensolutions and hence may be constructed from Eq. (65) and (66), subject to condition (64). Two cases are significant: (1) $\nu = 1/2$, $n = 0$, and (2) $\nu = 1$, $n = 1$. The first case gives the leading term in the inner expansion of the axial velocity, Eq. (67a); all other terms are eliminated by matching. The second case gives the leading term of the cross-flow, Eq. (67b); the corresponding pressure term is zero by matching. Note that the coefficients orthogonal to Eq. (68) vanish as a result of the orientation of the coordinate system.

The term $\mathbf{q}_2$ of the outer expansion (cf. Eq. 15a) consists of two parts:

$$\mathbf{q}_2 = \tilde{\nabla}\,[\phi^0_2(\tilde{x},\tilde{\rho}) + \phi^1_2(\tilde{x},\tilde{\rho}) \sin\theta] \tag{69a}$$

where

$$\phi^0_2 = -\frac{a}{4\pi}\frac{1}{\tilde{r}}, \qquad \tilde{r} = (\tilde{x}^2 + \tilde{\rho}^2)^{1/2} \tag{69b}$$

$$\phi^1_2 = \frac{b}{4\pi}\frac{\tilde{\rho}}{(\tilde{r} - \tilde{x})\tilde{r}} \tag{69c}$$

The potential ϕ_2^0 is required by $u_{1/2}^0$, as explained in Sections III-C and III-D. The remaining term in Eq. (69b) is the potential of a "horseshoe vortex" extending downstream from the origin along the positive $\tilde{x}$-axis. The horseshoe vortex term in the outer expansion is required by *matching*, as can be seen from the outer expansion of $\mathbf{q}_1^+$.

E. A Remark Concerning the Nonlinear Effect

We have remarked above (Section IV-C) that the order and degree of the coefficients in the Fourier expansion of the eigensolutions of Eq. (58) are subject to the condition

$$\nu - \frac{n}{2} - \frac{1}{2} = 0, 1, 2, \cdots \tag{70}$$

where n may be any non-negative integer. It appears also that the *forcing terms* in Eq. (62) vanish unless Eq. (70) is satisfied. This places an upper limit of $N = 2\nu - 1$ on the degree of the coefficients of order ν in the Fourier expansion of an inner term (cf. Eq. 61). Thus, coefficients of large degree are necessarily of large order.

In order to prove the last assertion, it is convenient to introduce the notation (ν, n) for a term of order ϵ^ν in the inner expansion (56) whose Fourier expansion involves a nonzero coefficient of degree $n (n = 0, 1, 2, \cdots)$.[2] The forcing terms in Eq. (58) may be divided into linear and nonlinear parts. As a result of the linear parts, (a, b) generates a higher-order term $(a + 1, b)$. The nonlinear parts of the forcing terms, however, may generate higher or lower harmonics. In particular, (a, b) and (c, d) may interact nonlinearly to generate any of the terms[3]

[2] The discussion of the present section may easily be extended to include terms of intermediate order, e.g., of order $\epsilon^\nu (\log \epsilon)^\mu$, $\mu > 0$.

[3] Note that, as a result of our choice of dependent variables, a term (ν, n) may generate a "higher-order" term of the same formal order. The question here is entirely one of numbering the terms in a manner consistent with the possibility of crossflow. In fact, no such terms appear in the expansions of the crossflow.

$$\left(a + c + \frac{1}{2} , c + d \right)$$

$$\left(a + c - \frac{1}{2} , c + d \right)$$

$$\left(a + c + \frac{1}{2} , |c - d| \right)$$

or

$$\left(a + c - \frac{1}{2} , |c - d| \right)$$

A simple calculation shows that, whenever (a, b) and (c, d) satisfy Eq. (70), so do all terms which they may generate in the higher-order computations. Since the leading terms are eigensolutions, the assertion follows by induction.

Table 1 shows several terms of integral order which arise as a result of the nonlinear effect. The symbol $(a, b):(c, d)$ denotes a term generated by nonlinear interaction of (a, b) and (c, d).

Table 1. Integral orders in the inner expansion

ν \ n	0	1	2	3
1/2	Leading terms due to drag			
1		Leading terms due to lift (1/2, 0) : (1, 1)		
3/2	(1/2, 0) : (3/2, 0) (1, 1) : (1, 1)		(1/2, 0) : (3/2, 2) (1, 1) : (1, 1)	
2		(1/2, 0) : (1, 1) (3/2, 0) : (1, 1) (3/2, 2) : (1, 1)		(1/2, 0) : (2, 3) (3/2, 2) : (1, 1)

It can be shown that, to compute the entry $\nu = a$, $n = b$ in Table 1, it is necessary to consider only the terms which appear on or within the upward-running diagonals from the point $\nu = a$, $n = b$. Similarly, one finds that the "domain of influence" of a term (a, b) consists of the entries on or within the downward-running diagonals.

F. The First Effect of Wake Displacement

The Fourier expansion of the term of order ϵ in the inner expansion of the axial velocity is

$$u_1 = u_1^1(\bar{x}, \bar{\rho}) \sin\theta + \underline{u}_1^1(\bar{x}, \bar{\rho}) \cos\theta \tag{71}$$

The coefficients in Eq. (71) satisfy (cf. Eq. 62a)

$$H_1(u_1^1) + \frac{\partial p_1^1}{\partial \bar{x}} = f_1^1 \tag{72a}$$

$$H_1(\underline{u}_1^1) + \frac{\partial \underline{p}_1^1}{\partial \bar{x}} = \underline{f}_1^1 \tag{72b}$$

The equation for pressure is homogeneous, and, by matching (cf. Eq. A-14, Appendix A),

$$p_1^1 = \underline{p}_1^1 = 0 \tag{73}$$

The forcing terms in Eq. (72) are

$$f_1^1 = -v_1^1 \frac{\partial u_{1/2}^0}{\partial \bar{\rho}}$$

$$= \frac{\alpha\omega}{\bar{\rho}\,\bar{x}^2}\left(e^{-\sigma} - e^{-2\sigma}\right) \tag{74a}$$

$$\underline{f}_1^1 = 0 \tag{74b}$$

where

$$\alpha = \frac{a\mathfrak{R}_e}{4\pi}, \qquad \omega = \frac{b\mathfrak{R}_e}{4\pi} \tag{75}$$

One finds

$$u_1^1 = \frac{1}{2}\,\alpha\omega\,\frac{\sigma}{\bar{\rho}\,\bar{x}}\left[e^{-\sigma}\,\mathrm{Ei}(-\sigma) + \frac{1}{\sigma}\left(e^{-2\sigma} - e^{-\sigma}\right) - e^{-\sigma}\log\sigma + (\log\bar{x} - \gamma + 1)\,e^{-\sigma}\right] + u_{1_e}^1 \tag{76a}$$

$$\underline{u}_1^1 = \underline{u}_{1_e}^1 \tag{76b}$$

Equation (76a) represents perturbation of the flow due to "displacement of the wake." The nonlinear interaction is between the leading terms due to drag and to lift. The downwash field associated with the trailing vortex displaces the wake below the positive $\bar{x}$-axis. The first correction for this effect appears in the term u_1. (The crossflow associated with the deflection of the wake is of higher order, of order ϵ^2, cf. Section IV-H.)

The switchback term u_{1a}^1 may be obtained from Eq. (76a) in the usual way. The appearance of a term analogous to u_{1a}^1 in the expansion for the two-dimensional case is linked historically with the so-called "Filon Paradox" (see Ref. 2; see also Section IV-L).

G. The Crossflow of Order $\epsilon^{3/2}$

The terms of order $\epsilon^{3/2}$ in the inner expansion of the crossflow components and pressure have the representations:

$$v_{3/2} = v_{3/2}^0(\bar{x},\bar{\rho}) + v_{3/2}^2(\bar{x},\bar{\rho})\sin 2\theta + \underline{v}_{3/2}^2(\bar{x},\bar{\rho})\cos 2\theta \tag{77a}$$

$$w_{3/2} = w_{3/2}^0(\bar{x},\bar{\rho}) + w_{3/2}^2(\bar{x},\bar{\rho})\sin 2\theta + \underline{w}_{3/2}^2(\bar{x},\bar{\rho})\cos 2\theta \tag{77b}$$

$$p_{3/2} = p_{3/2}^0(\bar{x},\bar{\rho}) + \underline{p}_{3/2}^2(\bar{x},\bar{\rho})\cos 2\theta \tag{77c}$$

The coefficients satisfy the following systems of equations:

$$H_1(w^0_{3/2}) = 0 \tag{78a}$$

$$\frac{\partial}{\partial \bar{\rho}} (\bar{\rho}\, v^0_{3/2}) = h^0_{3/2} \tag{78b}$$

$$L_0(p^0_{3/2}) = k^0_{3/2} \tag{79}$$

$$H_2(\bar{\rho}\, v^2_{3/2}) + \bar{\rho} \frac{\partial}{\partial \bar{\rho}} p^2_{3/2} = 0 \tag{80a}$$

$$\frac{\partial}{\partial \bar{p}} (\bar{\rho}\, v^2_{3/2}) - 2\, \underline{w}^2_{3/2} = 0 \tag{80b}$$

$$L_2(p^2_{3/2}) = 0 \tag{80c}$$

$$H_2(\bar{\rho}\, \underline{v}^2_{3/2}) + \bar{\rho} \frac{\partial}{\partial \bar{\rho}} \underline{p}^2_{3/2} = \underline{g}^2_{3/2} \tag{81a}$$

$$\frac{\partial}{\partial \bar{\rho}} (\bar{\rho}\, \underline{v}^2_{3/2}) + 2 w^2_{3/2} = 0 \tag{81b}$$

$$L_2(\underline{p}^2_{3/2}) = \underline{k}^2_{3/2} \tag{81c}$$

where

$$h^0_{3/2} = - \frac{\partial}{\partial \bar{x}} (\bar{\rho}\, u^0_{1/2} \tag{82a}$$

$$k^0_{3/2} = \frac{\partial v^1_1}{\partial \bar{\rho}} \left(\frac{\partial \underline{w}^1_1}{\partial \bar{\rho}} - \frac{\partial v^1_1}{\partial \bar{\rho}} \right) \tag{82b}$$

$$\underline{g}^2_{3/2} = \frac{1}{2}\left[(\underline{w}^1_1)^2 - (v^1_1)^2\right] \tag{82c}$$

$$\underline{k}^2_{3/2} = -\frac{\partial}{\partial\overline{\rho}}(\underline{g}^2_{3/2}) - \left[\frac{1}{\overline{\rho}}(\underline{w}^1_1)^2 - v^1_1\frac{\partial}{\partial\overline{\rho}}\underline{w}^1_1 - \frac{1}{\overline{\rho}}v^1_1\underline{w}^1_1\right] \tag{82d}$$

and v^1_1 and $\underline{w}^1_1$ are given by Eq. (68b).

The solution of Eq. (78) is

$$v^0_{3/2} = -\frac{a}{2\pi}\frac{\sigma e^{-\sigma}}{\overline{\rho}\,\overline{x}}, \qquad w^0_{3/2} = -\frac{m\Re_e}{4\pi}\frac{\sigma e^{-\sigma}}{\overline{\rho}\,\overline{x}} \tag{83}$$

where m is an arbitrary constant, related to the axial torque. The terms $v^0_{3/2}$ and $w^0_{3/2}$ are therefore identical with the terms of equivalent order which were constructed in Part III (cf. Eq. 22).

Integrating Eq. (79), one finds

$$p^0_{3/2} = \frac{\omega^2}{4}\frac{1}{\overline{x}^2}\left[\mathrm{Ei}(-\sigma) - \mathrm{Ei}(-2\sigma) + \frac{1}{\sigma}(e^{-\sigma} - e^{-2\sigma}) + \frac{1}{2\sigma^2}(2e^{-\sigma} - e^{-2\sigma} - 1) - \frac{a}{4\pi}\frac{1}{\overline{x}^2}\right] \tag{84}$$

The last term on the right was obtained in Section III-D. Equation (84) shows that, if the lift is not zero, pressure does not penetrate the wake, even to the first approximation. The difference in pressure across the wake has the expansion

$$\overline{p}(\overline{x},\infty) - \overline{p}(\overline{x},0) = \epsilon^{3/2}\frac{\omega^2}{8\overline{x}^2}(2\log 2 - 1) + o(\epsilon^{3/2}) \tag{85}$$

The term exhibited on the right balances centrifugal forces within the trailing vortices, and hence represents a nonlinear effect.

The solutions of Eq. (80) and (81) are

$$v^2_{3/2} = v^2_{3/2_e} \tag{86a}$$

$$\underline{w}^2_{3/2} = \underline{w}^2_{3/2_e} \tag{86b}$$

$$p^2_{3/2} = 0 \tag{86c}$$

$$\begin{aligned} \underline{v}^2_{3/2} = \frac{2\omega^2}{\mathcal{R}_e} \frac{1}{\bar{\rho}^3} \Bigg\{ & \frac{\sigma^2}{2} \left[\mathrm{Ei}(-2\sigma) - \mathrm{Ei}(-\sigma)\right] + \frac{\sigma}{4} \left[e^{-2\sigma} - 2e^{-\sigma} - e^{-\sigma} \log \sigma + e^{-\sigma}\mathrm{Ei}(-\sigma)\right] \\ & + \frac{1}{8} \left[e^{-2\sigma} - 4e^{-\sigma} + 2e^{-\sigma}\,\mathrm{Ei}(-\sigma) - 2e^{-\sigma} \log \sigma - 2\mathrm{Ei}(-2\sigma) + 2\mathrm{Ei}(-\sigma)\right] \\ & + \frac{1}{4} (3/2 + \log 2 - \gamma)(\sigma + 1) e^{-\sigma} + \frac{1}{4} (\log \bar{x})(\sigma + 1) e^{-\sigma} - \frac{1}{4} \log \bar{x} \Bigg\} + \underline{v}^2_{3/2_e} \end{aligned} \tag{87a}$$

$$w^2_{3/2} = -\frac{1}{2} \frac{\partial}{\partial \bar{\rho}} (\bar{\rho}\, \underline{v}^2_{3/2}) \tag{87b}$$

$$\underline{p}^2_{3/2} = \frac{\omega^2}{4} \frac{1}{\bar{x}^2} \left[\sigma\,\mathrm{Ei}(-2\sigma) - \sigma\,\mathrm{Ei}(-\sigma) + \frac{1}{2}(e^{-2\sigma} - 2e^{-\sigma}) - \frac{1}{4\sigma}(3e^{-2\sigma} - 4e^{-\sigma}) - \frac{1}{4\sigma}\right] \tag{87c}$$

Equation (86c) follows from matching. From Eq. (87) one sees that $\underline{v}^2_{3/2}$ and $w^2_{3/2}$ are associated with switchback terms $\underline{v}^2_{3/2a}$ and $w^2_{3/2a}$. The switchback term in the inner expansion of the crossflow (cf. Eq. 56b) is then of the form

$$\mathbf{q}^+_{3/2a} = w^2_{3/2a}(\bar{x}, \bar{\rho}) \sin 2\theta + \underline{v}^2_{3/2a}(\bar{x}, \bar{\rho}) \cos 2\theta \tag{88}$$

H. Higher-Order Inner Terms Associated With Wake Displacement and Lift

An explicit and complete description of inner terms of higher order will not be given in this Report. However, it is possible to determine, from the results given above, the terms of order $\epsilon^3 \log \epsilon$ and of order ϵ^3 in the outer expansion (Eq. 53). The procedure is to find in turn all terms of the inner expansion which vanish algebraically as $\bar{x} \to 0$, and which, in particular, contain nontrivial terms of order $\epsilon^3 \log \epsilon$ or ϵ^3 when expressed in outer variables. The inner terms in question are most easily interpreted through the corresponding pressure terms. We note in passing, however, that the inner and outer expansions of the cross-flow velocity would serve our purpose equally well, the equivalence being implied at each step by the matching conditions imposed on a given partial sum of the series.

It follows from the structure of the inner expansion of the pressure, and by matching, that inner pressure terms which are $O(\epsilon^3)$ in outer variables and which are not eigensolutions occur only when (ν, n) has the values (3/2, 2), (2, 1), or (5/2, 0), i.e., when $\nu + n/2 = 5/2$. (We include at the same time all associated switchback terms. Note that eigensolutions in the pressure which are $O(\epsilon^3)$ in outer variables occur in terms of arbitrarily large order and degree, subject only to condition 70). The equation for the pressure terms, obtained by taking the divergence of Eq. (3a), becomes, in the variables (14), (54), and (55),

$$-\bar{\nabla}_+^2 \bar{p} = \epsilon^{-1/2}\, \bar{\nabla}_+ \mathbf{q}^+ : \bar{\nabla}_+ \mathbf{q}^+ + 2\epsilon^{1/2}\, \bar{\nabla}_+ \bar{u} \cdot \frac{\partial \mathbf{q}^+}{\partial \bar{x}} + \epsilon \frac{\partial^2 \bar{p}}{\partial \bar{x}^2} + \epsilon^{3/2} \left(\frac{\partial \bar{u}}{\partial \bar{x}}\right)^2 \tag{89}$$

The symbol (:) denotes here the scalar tensorial product. If the inner expansion (Eq. 56) is inserted into Eq. (89) one finds that there are (apart from switchback terms) four contributions to the right-hand side of Eq. (89) which were not considered previously, and which are associated with possible algebraic terms of the required type. (Discussion of contributions from the terms $(\nu, n) = (3/2, 2)$ is omitted here, since these terms were given explicitly in Section IV-G). We define the resulting pressure terms by $\bar{p}^{(i)}$, $i = 1, \cdots, 4$. Then

$$-\bar{\nabla}_+^2 \bar{p}^{(1)} = \frac{\partial^2 p_{3/2}}{\partial \bar{x}^2} + \frac{3a}{8\pi} \frac{1}{\bar{x}^4} \tag{90a}$$

$$-\bar{\nabla}_+^2 \bar{p}^{(2)} = 2\bar{\nabla}_+ u_{1/2} \cdot \frac{\partial \mathbf{q}_1^+}{\partial \bar{x}} + \bar{\nabla}_+ \mathbf{q}_1^+ : \bar{\nabla}_+ \mathbf{q}_{3/2}^0 + \bar{\nabla}_+ \mathbf{q}_{3/2}^0 : \bar{\nabla}_+ \mathbf{q}_1^+ \tag{90b}$$

$$-\bar{\nabla}_+^2 \bar{p}^{(3)} = -2\bar{\nabla}_+ u_1 \cdot \frac{\partial \mathbf{q}_1^+}{\partial \bar{x}} \tag{90c}$$

$$-\bar{\nabla}_+^2 \bar{p}^{(4)} = \bar{\nabla}_+(\mathbf{q}_{3/2}^+ - \mathbf{q}_{3/2}^0) : \bar{\nabla}_+ \mathbf{q}_1^+ + \bar{\nabla}_+ \mathbf{q}_1^+ : \bar{\nabla}_+(\mathbf{q}_{3/2}^+ - \mathbf{q}_{3/2}^0) \tag{90d}$$

The first two equations have been solved and the results that are needed later are contained in Eq. (93) and (94) below. The term $\bar{p}^{(3)}$ results from the nonlinear interaction between the axial velocity due to wake displacement, and the leading terms due to lift. There is a possible contribution to the outer expansion of p^* which is of order ϵ^3. It will be shown, however, that, with the possible exception of additive functions of $\bar{x}$, there is no algebraic term which is independent of θ and which has this property. It is sufficient to prove that the axially symmetric part of $\bar{p}^{(3)}$ is $o(\log \bar{\rho})$ as $\bar{\rho} \to \infty$. Now Eq. (90c) may be written as a conservation law in the crossflow plane

$$\bar{\nabla}_+ \cdot \left(\bar{\nabla}_+ \bar{p}^{(3)} + u_1 \frac{\partial \mathbf{q}_1^+}{\partial \bar{x}} \right) \tag{91}$$

since $\mathbf{q}_1^+$ is divergenceless in the crossflow plane. The assertion follows immediately by integration of Eq. (91) over the circle $\bar{\rho} = \bar{\rho}_0$ in the crossflow plane, using Eq. (67b), (68b), (71), and (77) to evaluate the resulting contour integral as $\bar{\rho}_0 \to \infty$.

By a similar device, it can be shown that $\bar{p}^{(4)}$ is $o(\bar{\rho}^{-1})$ as $\bar{\rho} \to \infty$, and hence no algebraic term of degree 1 and order ϵ^3 is contributed to the outer expansion of p^*. Here, use is made of the conservation law

$$\begin{aligned}
\bar{\nabla}_+ \cdot \{ & \bar{y}\,\bar{\nabla}_+ \bar{p}^{(4)} - \bar{p}^{(4)}\mathbf{j} + \bar{y}\,[\mathbf{q}_1^+ \cdot \nabla(\mathbf{q}_{3/2}^+ - \mathbf{q}_{3/2}^0) + (\mathbf{q}_{3/2}^+ - \mathbf{q}_{3/2}^0) \cdot \nabla \mathbf{q}_1^+] \\
& - 2(\mathbf{q}_{3/2}^+ \cdot \mathbf{j} - \mathbf{q}_{3/2}^0 \cdot \mathbf{j})(\mathbf{q}_1^+ \cdot \mathbf{j})\,\mathbf{j} - [(\mathbf{q}_{3/2}^+ \cdot \mathbf{k} - \mathbf{q}_{3/2}^0 \cdot \mathbf{k})(\mathbf{q}_1 \cdot \mathbf{j}) \\
& + (\mathbf{q}_{3/2}^+ \cdot \mathbf{j} - \mathbf{q}_{3/2}^0 \cdot \mathbf{j})(\mathbf{q}_1^+ \cdot \mathbf{k})]\,\mathbf{k}\} = 0
\end{aligned} \tag{92}$$

and an orthogonal law obtained by replacing $\bar{y}$ by $\bar{z}$ and interchanging $\mathbf{j}$ and $\mathbf{k}$ in Eq. (92).

The needed results concerning $\overline{p}^{(1)}$ and $\overline{p}^{(2)}$ may be stated as follows: There is a term $\epsilon^2 (p_2^1 \sin\theta + \underline{p}_2^1 \cos\theta)$ in the inner expansion of $\overline{p}$ which has the outer expansion

$$\epsilon^{5/2} (p_2^1 \sin\theta + \underline{p}_2^1 \cos\theta) = -\epsilon^{3/2} \frac{\omega}{\mathfrak{R}_e} \frac{\tilde{\rho}}{\tilde{x}^3} \sin\theta + \epsilon^{5/2} \frac{\alpha\omega}{\mathfrak{R}_e} \frac{\sin\theta}{\tilde{\rho}\tilde{x}^2} + o(\epsilon^{5/2}) \tag{93a}$$

Also, there is a term $\epsilon^{5/2} p_{5/2}^0$ in the inner expansion of $\overline{p}$ which has the expansion

$$\epsilon^{5/2} p_{5/2}^0 = \epsilon^{5/2} \log\epsilon \frac{c_3}{\tilde{x}^3} + \epsilon^{5/2} \left\{ \left(\frac{\omega^2}{\mathfrak{R}_e} - \frac{2\alpha^2}{\mathfrak{R}_e} \right) \frac{\log\tilde{\rho}}{\tilde{x}^3} + c_4 \frac{\log\tilde{x}}{\tilde{x}^3} + \frac{c_5}{\tilde{x}^3} \right\} + o(\epsilon^{5/2}) \tag{93b}$$

Moreover, Eq. (93a) and (93b) contain, apart from eigensolutions and terms of degree 2, the only pressure terms which match with the term of order ϵ^3 in the outer expansion of the pressure. We note that the first term on the right of Eq. (93a) matches with $\tilde{p}_2$ (cf. Eq. 53b). The constants c_4 and c_5 appearing in Eq. (93b) are determined at a later stage by matching; c_3 may subsequently be found by applying the eliminability principle.

I. The Outer Term of Order ϵ^3

The outer term $\mathbf{q}_3$ (cf. Eq. 53a) is related to a potential ϕ_3 by

$$\mathbf{q}_3 = \tilde{\nabla}\phi_3, \qquad \tilde{\nabla}^2 \phi_3 = 0 \tag{94a}$$

where, with $\mu = \tilde{x}/\tilde{r}$

$$\phi_3 = \phi_3^0(\tilde{r},\mu) + \phi_3^1(\tilde{r},\mu)\sin\theta + \underline{\phi}_3^1(\tilde{r},\mu)\cos\theta + \phi_3^2(\tilde{r},\mu)\sin 2\theta + \underline{\phi}_3^2(\tilde{r},\mu)\cos 2\theta \tag{94b}$$

The $\phi_\lambda^n(\tilde{r},\mu)$ are given by

$$\phi_3^0(\tilde{r},\mu) = \left(\frac{\omega^2}{4\Re_e} - \frac{\alpha^2}{2\Re_e}\right) C_1^0(\tilde{r},\mu) + \frac{1}{2}\,\tilde{a}_{5/2}^0\,\frac{\mu}{\tilde{r}^2} \tag{94c}$$

$$\phi_3^1(\tilde{r},\mu) = \frac{\alpha\omega}{2\Re_e}\,C_1^1(\tilde{r},\mu) + \frac{1}{3}\,\tilde{a}_3^1\,\frac{(1-\mu^2)^{1/2}}{\tilde{r}^2} \tag{94d}$$

$$\underline{\phi}_3^1(\tilde{r},\mu) = \frac{1}{3}\,\tilde{\underline{a}}_3^1\,\frac{(1-\mu^2)^{1/2}}{\tilde{r}^2} \tag{94e}$$

$$\phi_3^2(\tilde{r},\mu) = -b_{3/2}^2\,B_1^2(\tilde{r},\mu) \tag{94f}$$

$$\underline{\phi}_3^2(\tilde{r},\mu) = \frac{\omega^2}{4\Re_e}\,C_1^2(\tilde{r},\mu) - \underline{b}_{3/2}^2\,B_1^2(\tilde{r},\mu) \tag{94g}$$

The functions $B_\lambda^n(\tilde{r},\mu)$ and $C_\lambda^n(\tilde{r},\mu)$ are defined in Appendix A (see Section III). The switchback term $\mathbf{q}_{3a}$ may be constructed in the usual manner, using the definition of $C_\lambda^n(\tilde{r},\mu)$.

In addition to the eigensolutions of this order, the solutions C_1^0, C_1^1, and C_1^2 are required by matching. If the lift is zero, the axially symmetric term coincides with that given in Part III (cf. Eq. 42b). That part proportional to ω^2 is required to match the pressure to order ϵ^3 inclusive.[4] It is interesting to note that the *inflow* into the wake due to drag (see Section III-G) is now opposed by an *outflow* due to crossflow. The two effects balance exactly when $b = \sqrt{2}\,a$. The solution C_1^1 is associated with wake deflection, and the corresponding pressure matches with the second term on the right of Eq. (93a). Finally, C_1^2 is required to match the two expansions of r^* to order $\epsilon^{3/2}$ inclusive. This can be seen from the outer expansion

$$\underline{v}_{3/2}^2 \sim -\frac{\omega^2}{2\Re_e}\,\frac{\log \tilde{x}}{\tilde{\rho}^3}\,\epsilon^{3/2} \tag{95}$$

for the crossflow coefficient determined in Section IV-G.

[4] Note that the terms of the outer expansion of velocity and pressure (Eq. 53) are here related by Eq. (18), Part III.

J. The Composite Expansion

The composite expansion for the multidimensional problem may be obtained by the same procedures used in the axially symmetric and two-dimensional cases. It appears that the number of steps required to reach a given degree of approximation is far greater in the lifting case, owing primarily to the complicated interaction among the crossflow terms. However, the essentially second-order terms given above suffice to form an approximation valid uniformly to order $\epsilon^{3/2}$ in the velocity and order ϵ^2 in the pressure as $\epsilon \to 0$, or, equivalently, to the respective orders $r^{*-3/2}$ and r^{*-2} as $r^* \to \infty$ (cf. Appendix B, Section II). Note that outer terms which are $o(\epsilon^2)$ will in no way be involved in this approximation.

Proceeding as in Section III-J, we define

$$\widetilde{E}_2(\mathbf{q}^*) = \epsilon^2 \mathbf{q}_2(\widetilde{x}_i) = \mathbf{q}_o^*(x_i^*) \tag{96a}$$

$$\widetilde{E}_2(p^*) = \epsilon^2 p_2(\widetilde{x}_i) = p_o^*(x_i^*) \tag{96b}$$

$$\overline{E}_{3/2}(\mathbf{q}^+) = \mathbf{q}_i^+(x_i^*) \tag{96c}$$

$$1 + \epsilon^{1/2}\overline{E}_1(\overline{u}) = u_i^*(x_i^*) \tag{96d}$$

$$\epsilon^{1/2}\overline{E}_{3/2}(\overline{p}) = p_i^*(x_i^*) \tag{96e}$$

$$u_c^* = 0 \tag{97a}$$

$$v_c^* = -\frac{b}{2\pi}\frac{\sin\theta}{\rho^{*2}} + 2b_{3/2}^2\frac{\sin 2\theta}{\rho^{*3}} - \frac{\omega^2}{2\mathcal{R}e}\frac{\log x^*\cos 2\theta}{\rho^{*3}} + 2\underline{b}_{3/2}^2\frac{\cos 2\theta}{\rho^{*3}} \tag{97b}$$

$$w_c^* = \frac{b}{2\pi}\frac{\cos\theta}{\rho^{*2}} - 2b_{3/2}^2\frac{\cos 2\theta}{\rho^{*3}} - \frac{\omega^2}{2\mathcal{R}e}\frac{\log x^*\sin 2\theta}{\rho^{*3}} + 2\underline{b}_{3/2}^2\frac{\sin 2\theta}{\rho^{*3}} \tag{97c}$$

$$p_c^* = -\frac{a}{4\pi}\frac{1}{x^{*2}} \tag{97d}$$

$$\mathbf{q}_c^* = u_c^* \, \mathbf{i}_x + v_c^* \, \mathbf{i}_\rho + w_c^* \, \mathbf{i}_\theta \tag{98a}$$

$$\mathbf{q}_i^* = u_i^* \, \mathbf{i}_x + \mathbf{q}_i^\dagger \tag{98b}$$

With these definitions, the desired composite expansion is again given by Eq. (52), and is valid to the respective orders stated above.

The most surprising new feature of the general three-dimensional solutions occurs in the first approximation to the pressure. It is clear from the results of Section IV-G that the nonlinear effect manifests itself in the leading term of the series for the pressure. To put this differently, it is impossible to construct the first term in the Navier-Stokes series for the pressure, purely from solutions of the linear Oseen equations. As it happens, however, the order of the leading term is in either case the same.

K. The Calculation of Force and Moment

The expansions (53) and (56) allow a determination of the force and moment which act upon the solid (or upon any closed streamsurface containing the solid) in terms of the arbitrary constants which occur in the series. It should be emphasized that these dynamical relations imply no constraints on the constants themselves and serve rather to relate them to observable physical quantities.

The conservation laws for momentum and for angular momentum are defined by

$$\nabla^* \cdot \underline{\underline{A}}^* = 0 \tag{99a}$$

$$\nabla^* \cdot \underline{\underline{M}}^* = 0 \tag{99b}$$

where the tensors $\underline{\underline{A}}^*$ and $\underline{\underline{M}}^*$ are given by

$$\underline{\underline{A}}^* = \mathbf{q}^* \circ \mathbf{q}^* + p^* \underline{\underline{I}} - \frac{1}{\mathcal{R}_e} \operatorname{def} \mathbf{q}^* \tag{100a}$$

$$\underline{\underline{M}}^* = \mathbf{r}^* \times \underline{\underline{A}}^* \tag{100b}$$

Here,

$$\underline{\underline{I}} = \text{identity tensor and def } \mathbf{q}^* = \text{deformation tensor.}$$

If Gauss' theorem is applied to Eq. (99), the surface integral will consist of two parts. The first may be chosen to be the body surface S_0; the second will be chosen to be a sphere of radius R_0 which contains S_0.

In the limit $R_0 \to \infty$, the asymptotic expansions may be used to evaluate the integral over the latter. One finds

$$\mathbf{F}^* = \iint_{S_0} \underline{\underline{A}}^* \cdot \mathbf{ds}^* = a\,\mathbf{i} + b\,\mathbf{j} \tag{101a}$$

$$\mathbf{M}^* = \iint_{S_0} \underline{\underline{M}}^* \cdot \mathbf{ds}^* = m\,\mathbf{i} + \left(\frac{4\pi}{\mathfrak{Re}}\,\underline{a}_1^1 - \frac{4}{3}\,\pi\,\underline{\tilde{a}}_3^1 \right)\mathbf{i}$$
$$+ \left[\frac{4\pi\,a_1^1}{\mathfrak{Re}} - \frac{4}{3}\,\pi\,\tilde{a}_3^1 + \frac{2b}{\mathfrak{Re}} - \frac{4\pi\,a\omega}{\mathfrak{Re}} \right]\mathbf{k} \tag{101b}$$

where $\mathbf{ds}^*$ is the dimensionless area element, directed toward the interior of the body, and γ = Euler's constant. The terms on the left of Eq. (101a) and (101b) are respectively the dimensionless force and moment experienced by the solid.

The manner in which Eq. (101) is obtained will be illustrated by carrying out in detail the calculation of the $\mathbf{k}$ contribution to the moment. Consider first the contributions to this component from the inner expansions. We may assume the region of integration to consist of a portion of the crossflow plane $x^* = R_0^*$ whose area is large compared with ϵ^{-1}; we denote this region by Σ_i^*. Now the integrand under consideration becomes, in cylindrical polar coordinates (Eq. 12),

$$\mathbf{k} \cdot (\underline{\underline{M}}^* \cdot \mathbf{i}) = x^* u^* v^* \sin\theta + x^* u^* w^* \cos\theta - \rho^* u^{*2} \sin\theta - \rho^* p^* \sin\theta - \frac{x^*}{\mathfrak{Re}} \frac{\partial v^*}{\partial x^*} \sin\theta$$

$$- \frac{x^*}{\mathfrak{Re}} \frac{\partial w^*}{\partial x^*} \cos\theta - \frac{x^*}{\mathfrak{Re}} \frac{\partial u^*}{\partial \rho^*} \sin\theta - \frac{x^*}{\mathfrak{Re}\,\rho^*} \frac{\partial u^*}{\partial \theta} \cos\theta + \frac{2\rho^*}{\mathfrak{Re}} \sin\theta \frac{\partial u^*}{\partial x^*} \tag{102}$$

Expanding the right-hand side of Eq. (102) in inner variables, using Eq. (56) as well as the Fourier expansions for each inner term, we then have

$$\begin{aligned}\mathbf{k}^* \cdot (\underline{\underline{M}}^* \cdot \mathbf{i}) = {} & \bar{x}\, v_1^1 \sin^2\theta + \bar{x}\, \underline{w}_1^1 \cos^2\theta + \epsilon\, \bar{x}\, u_{1/2}^0\, v_1^1 \sin^2\theta \\ & + \epsilon\, \bar{x}\, u_{1/2}^0\, \underline{w}_1^1 \cos^2\theta + \epsilon\, \bar{x}\, v_{(2)}^1 \sin^2\theta + \epsilon\, \bar{x}\, w_{(2)}^1 \cos^2\theta \\ & - 2\epsilon\, \bar{\rho}\, u_{(1)}^1 \sin^2\theta - \epsilon \frac{\bar{x}}{\mathfrak{Re}} \frac{\partial v_1^1}{\partial \bar{x}} \sin^2\theta \\ & - \epsilon \frac{\bar{x}}{\mathfrak{Re}} \frac{\partial \underline{w}_1^1}{\partial \bar{x}} \cos^2\theta - \epsilon \frac{\bar{x}}{\mathfrak{Re}} \frac{\partial u_{(1)}^1}{\partial \bar{\rho}} \sin^2\theta \\ & - \epsilon \frac{\bar{x}}{\mathfrak{Re}\,\bar{\rho}}\, u_{(1)}^1 \cos^2\theta + \frac{1}{\epsilon}\, \mathcal{E}_2^1 \end{aligned} \tag{103}$$

Here $\mathcal{E}_2^1$ denotes any remainder which consists of terms which are either $o(\epsilon^2)$ or which vanish when integrated with respect to θ between the limits 0 and 2π. The terms $u_{(1)}^1(\bar{x}, \bar{\rho})$, $v_{(2)}^1(\bar{x}, \bar{\rho})$, and $w_{(2)}^1(\bar{x}, \bar{\rho})$ are not necessarily independent of ϵ and are defined by

$$\sin\theta\, v^* = \epsilon \sin^2\theta\, v_1^1 + \epsilon^2 \sin^2\theta\, v_{(2)}^1 + \mathcal{E}_1^2 \tag{104a}$$

$$\cos\theta\, w^* = \epsilon \cos^2\theta\, \underline{w}_1^1 + \epsilon^2 \cos^2\theta\, \underline{w}_{(2)}^1 + \mathcal{E}_1^2 \tag{104b}$$

Also, we note that

$$\frac{1}{\bar{\rho}} \frac{\partial [\bar{\rho}\, v_{(2)}^1]}{\partial \bar{\rho}} - \frac{1}{\bar{\rho}}\, \underline{w}_{(2)}^1 = - \frac{\partial u_{(1)}^1}{\partial \bar{x}} \tag{105a}$$

$$\lim_{\bar{\rho} \to \infty} \bar{x}\bar{\rho}^2 v^1_{(2)} = -\frac{\alpha\omega}{\mathfrak{Re}} \tag{105b}$$

The last expression is actually equivalent to a matching condition on the crossflow velocity.

It follows from Eq. (103), (104), and (105) that

$$\mathbf{k} \cdot \iint_{\Sigma_i} \underline{\underline{M}}^* \cdot \mathbf{ds}^* = \pi \int_0^\infty \left\{ \frac{\bar{x}}{\epsilon} (v^1_1 + \underline{w}^1_1) + \bar{x} u^0_{1/2} (v^1_1 + \underline{w}^1_1) - 2\rho u^1_{(1)} - \frac{\bar{x}}{\mathfrak{Re}} \frac{\partial}{\partial \bar{x}} (v^1_1 + \underline{w}^1_1) \right.$$

$$\left. + \bar{\rho}\bar{x} \frac{\partial u^1_{(1)}}{\partial \bar{x}} - \frac{\bar{x} u^1_{(1)}}{\mathfrak{Re}\,\bar{\rho}} \right\} \bar{\rho}\, d\bar{\rho} - \frac{\pi \alpha\omega}{\mathfrak{Re}} + o(1) = \bar{M}_3 \tag{106}$$

Evaluating these integrals, and setting the parameter $\bar{x} = \epsilon R_0$, one obtains

$$\bar{M}_3 = -\frac{2\pi\omega}{\mathfrak{Re}} R^*_0 - \frac{2\pi\alpha\omega}{\mathfrak{Re}} \log R^*_0 + \frac{2\pi\alpha\omega}{\mathfrak{Re}} \left(\log 2 + \frac{1}{2}\right) - \frac{4\pi}{\mathfrak{Re}} a^1_1 \tag{107}$$

To compute the outer contribution, we may assume that $\mathbf{q}^*$ possesses a potential $\phi^*(\tilde{x}_i; \epsilon)$. Define, for $\cos\mu = \psi$, $\tilde{r}$ fixed,

$$\int_0^{2\pi} \phi^* \sin\theta\, d\theta = \epsilon^2 F_2(\tilde{r},\psi) + \epsilon^3 \log\epsilon\; F_{3a}(\tilde{r},\psi) + \epsilon^3 F_3(\tilde{r},\psi) + o(\epsilon^3) \tag{108}$$

Also, let Σ_0 be the surface of the sphere $\tilde{r} = \epsilon R^*_0$ minus a region of area $O(\epsilon^2)$ containing the intersection with the positive $\tilde{x}$-axis. It then may be shown that

$$\mathbf{k} \cdot \iint_{\Sigma_0} \underline{\underline{M}}^* \cdot \mathbf{ds}^* = \frac{1}{\epsilon} I_2 + \log\epsilon\; I_{3a} + I_3 + I' + o(1)$$

$$= \tilde{M}_3 \tag{109a}$$

where

$$I_\nu = \int_{\tilde{r}=\epsilon R_0^*}^{\pi} \left\{ \tilde{r}^2 \cos\psi \sin\psi \frac{\partial F_\nu}{\partial \psi} + \tilde{r}^2 \cos^2\psi\, F_\nu - \tilde{r}^3 \sin^2\psi \frac{\partial F_\nu}{\partial \tilde{r}} \right\} d\psi \tag{109b}$$

$$I' = -\frac{2}{\mathfrak{R}_e} \int_{\tilde{r}=\epsilon R_0^*}^{\pi} \tilde{r}^3 \frac{\partial}{\partial \tilde{r}} \left\{ \frac{\sin\psi}{\tilde{r}} \frac{\partial F_2}{\partial \psi} + \frac{\cos\psi}{\tilde{r}} F_2 \right\} d\psi \tag{109c}$$

Note that I' is the contribution from the viscous stresses associated with the leading outer term due to lift.

From the expressions given in Sections IV-D and IV-I and in Appendix A, we have

$$F_2 = \frac{\omega\pi}{\mathfrak{R}_e} \frac{\sin\psi}{1-\cos\psi} \frac{1}{\tilde{r}} \tag{110a}$$

$$F_{3a} = -\frac{\alpha\omega\pi}{2\mathfrak{R}_e} \frac{\sin\psi}{\tilde{r}^2} \tag{110b}$$

$$F_3 = \frac{\alpha\omega\pi}{2\mathfrak{R}_e} \left\{ \sin\psi \frac{\log\tilde{r}}{\tilde{r}^2} + \left[\frac{\sin\psi}{1-\cos\psi} - \sin\psi \log(1-\cos\psi) \right] \frac{1}{\tilde{r}^2} \right\} + \frac{\pi}{3} \tilde{a}_3^1 \frac{\sin\psi}{\tilde{r}^2} \tag{110c}$$

A straightforward calculation then gives

$$I_2 = \frac{2\pi\omega}{\mathfrak{R}_e} R_0^* \epsilon \tag{111a}$$

$$I_{3a} = -\frac{2\alpha\omega\pi}{\mathfrak{R}_e} \tag{111b}$$

$$I_3 = \frac{2\alpha\omega\pi}{\mathfrak{R}_e} \log\epsilon R_0^* + \frac{3\alpha\omega\pi}{\mathfrak{R}_e} - \frac{2\alpha\omega\pi}{\mathfrak{R}_e} \log 2 + \frac{4\pi}{3} \tilde{a}_3^1 \tag{111c}$$

$$I' = -\frac{8\pi\omega}{\mathfrak{R}_e^2} = -\frac{2b}{\mathfrak{R}_e} \tag{111d}$$

Returning to Eq. (108), the outer contribution is then

$$\widetilde{M}_3 = \frac{2\pi\omega}{\mathcal{R}_e} R_0^* + \frac{2\pi\alpha\omega}{\mathcal{R}_e} \log R_0^* - \frac{2b}{\mathcal{R}_e} + \frac{3\alpha\omega\pi}{\mathcal{R}_e} - \frac{2\alpha\omega\pi}{\mathcal{R}_e} \log 2 + \frac{4\pi}{3} \tilde{a}_3^1 \qquad (112)$$

The sum of all contributions, inner and outer, thus gives

$$\mathbf{k} \cdot \iint_{S_0} \underline{\underline{M}}^* \cdot \mathbf{ds}^* = -(\overline{M}_3 + \widetilde{M}_3) = \frac{2b}{\mathcal{R}_e} - \frac{4\alpha\omega\pi}{\mathcal{R}_e} + \frac{4\pi}{\mathcal{R}_e} a_1^1 - \frac{4\pi}{3} \tilde{a}_3^1 \qquad (113)$$

which verifies the last term on the right of Eq. (101b).

L. Discussion

The principal conclusion that may be drawn from the results of this Part is that no fundamentally new ideas are needed to extend existing asymptotic procedures to the three-dimensional problem treated in this Report. However, as noted in Section IV-A, the presence of crossflow does significantly alter the form of the construction. The presence of crossflow is largely associated with the existence of a lift; in fact, the crossflow terms considered in this Report, apart from the eigensolutions, result entirely from the presence of the so-called "horseshoe vortex" terms of the inner and outer expansions (cf. Eq. 68b, 69c). Since these terms have no counterparts in the two-dimensional series, the effect of lift is different in the two problems.

The special nature of the lifting case considered in this Report is emphasized by the observation that the first approximation to the pressure is different for Navier-Stokes and Oseen solutions (see Section IV-J). No similar conclusion may be drawn from the two-dimensional or axially symmetric series. This point probably bears directly on the rigorous proof of statement 1 (Section II-B) for the general three-dimensional case.

Apart from these differences in form, the series described here are closely related to the results of Imai (Ref. 1) and Chang (Ref. 2). In particular, the orders which occur in both expansions appear to be of the form $\epsilon^{i/2} (\log \epsilon)^j$ where i and j are non-negative integers and j has an upper bound depending only upon i.[5]

[5] For the terms considered here, $j \leq i - 1$.

It is also interesting to note that the classical error leading to Filon's paradox has a direct three-dimensional counterpart. This is obvious from the presence of canceling terms in Eq. (107) and (112) which are individually of order log r^* as $r^* \to \infty$. The explanations of Filon's result that have been given by Imai and by Chang carry over with no essential changes to the present problem.

The basic equivalence of our parameter procedure to related coordinate-type procedures is apparent at any stage of the construction, and, with due regard to the occurrence of switchback, our results would indicate that there are no intrinsic advantages associated with either approach. A somewhat different question concerns the possibility of a more direct analysis in the crossflow plane. It may actually be preferable to integrate the primary crossflow system directly (Eq. 58b and 58c) rather than to start from the secondary (and derived) system (Eq. 62b, 62c, and 62d). The possible advantages of this modification have not, however, been studied in detail.

NOMENCLATURE

A. Variables and Parameters

L	reference length
R	extraneous length
ϵ	small parameter $= L/R$
U	reference velocity
P	reference pressure
ρ	density (constant)
ν	kinematic viscosity (constant)
$\mathfrak{Re}$	Reynolds number $= UL/\nu$
$\mathbf{r}$	position vector $= L\mathbf{r}^*$
$\mathbf{q}$	velocity $= U\mathbf{q}^*$
p	pressure $= \rho U^2 \mathrm{p}^* + P$
ϕ	velocity potential $= UL\phi^*$
$\underline{\underline{I}}$	identity tensor
def $\mathbf{q}$	deformation tensor, $(\mathrm{def}\,\mathbf{q})_{ij} = \dfrac{\partial q_i}{\partial x_j} + \dfrac{\partial q_j}{\partial x_i}$
$\underline{\underline{A}}$	$\rho\,\mathbf{q} \circ \mathbf{q} + (\mathrm{p} - P)\underline{\underline{I}} - \rho\nu\,\mathrm{def}\,\mathbf{q} = \rho U^2 \underline{\underline{A}}^*$
$\underline{\underline{M}}$	$\mathbf{r} \times \underline{\underline{A}} = \rho L U^2 \underline{\underline{M}}^*$
$\mathbf{F}$	force exerted on obstacle $= \rho U^2 L^2 \mathbf{F}^*$
$\mathbf{M}$	moment exerted on obstacle (positive by right-hand rule) $= \rho U^2 L^3 \mathbf{M}^*$

NOMENCLATURE (Cont'd)

B. Coordinate Systems

Cartesian: $\mathbf{i}, \mathbf{j}, \mathbf{k}$ = unit vectors

$\mathbf{r} = x_1 \mathbf{i} + x_2 \mathbf{j} + x_3 \mathbf{k}, \; x_1 = x, \; x_2 = y, \; x_3 = z$

$r = (x_1^2 + x_2^2 + x_3^2)^{1/2}$

Cylindrical Polar: $\mathbf{i}_x = \mathbf{i}, \; \mathbf{i}_\rho, \; \mathbf{i}_\theta$ = unit vectors

$\mathbf{r} = x\mathbf{i}_x + \rho\,\mathbf{i}_\rho$

$\rho = (y^2 + z^2)^{1/2}, \; \theta = \tan^{-1}\left(-\dfrac{y}{z}\right)$

$\mathbf{q} = u\,\mathbf{i}_x + v\,\mathbf{i}_\rho + w\,\mathbf{i}_\theta$

Spherical Polar: $\mu = x/r, \; \psi = \cos\mu$

$\mathbf{q} = U\mathbf{i}$ at infinity

$$a = \mathbf{F}^* \cdot \mathbf{i} = \text{drag}/\rho U^2 L^2 = \frac{4\pi\alpha}{\mathcal{R}_e}$$

$$b = \mathbf{F}^* \cdot \mathbf{j} = \text{lift}/\rho U^2 L^2 = \frac{4\pi\omega}{\mathcal{R}_e}$$

$$m = \mathbf{M}^* \cdot \mathbf{i} = \text{torque}/\rho U^2 L^3 = \frac{4\pi\beta}{\mathcal{R}_e}$$

C. The Outer Expansion

$$\tilde{x}_i = \epsilon x_i^*, \; \tilde{\nabla} = \frac{\partial}{\partial \tilde{x}_i}$$

$\mathbf{q}_\nu(\tilde{x}_i)$ = term in outer expansion of velocity

$\tilde{p}_\nu(\tilde{x}_i)$ = term in outer expansion of pressure

$\phi_\nu(\tilde{x}_i)$ = potential of $\mathbf{q}_\nu$, $\mathbf{q}_\nu = \tilde{\nabla}\phi_\nu$

$(\phi_\nu^n, \underline{\phi}_\nu^n)$ = coefficient of $(\sin n\theta, \; \cos n\theta)$ in Fourier expansion of ϕ_ν

NOMENCLATURE (Cont'd)

D. The Inner Expansion

$$\bar{x} = \epsilon x^*, \; \bar{y} = \epsilon^{1/2} y^* = \epsilon^{-1/2}\tilde{y}, \; \bar{z} = \epsilon^{1/2} z^* = \epsilon^{-1/2}\tilde{z}$$

$$\bar{\rho} = \epsilon^{1/2}\rho^* = \epsilon^{-1/2}\tilde{\rho}; \; \sigma = \frac{\Re_e \bar{\rho}^2}{4\bar{x}} = \frac{\Re_e \rho^{*2}}{4x^*} = \frac{\Re_e \tilde{\rho}^2}{4\tilde{x}\epsilon}$$

$$\bar{u} = \epsilon^{-1/2}(u^* - 1), \; \bar{v} = \epsilon^{-1/2} v^*, \; \bar{w} = \epsilon^{-1/2} w^*, \; \bar{p} = \epsilon^{-1/2} p^*$$

$$\mathbf{q}^+ = v^* \mathbf{i}_\rho + w^* \mathbf{i}_\theta, \; \bar{\nabla}_+ = \left(0, \frac{\partial}{\partial \bar{y}}, \frac{\partial}{\partial \bar{z}}\right)$$

u_ν = term in inner expansion of u^* (Part III) or $\bar{u}$ (Part IV)

(v_ν, w_ν) = term in inner expansion of $(\bar{v}, \bar{w})$ (Part III) or (v^*, w^*) (Part IV)

$(F_\nu^n, \underline{F}_\nu^n)$ = coefficient of $(\sin n\theta, \cos n\theta)$ in Fourier expansion of F_ν, $F = u, v, w, p$ (Part IV)

$F_{\nu_e}^n, \underline{F}_{\nu_e}^n$ = Fourier coefficients of eigensolutions (Part IV)

$a_1, m_1, a_\lambda^n, \tilde{a}_\lambda^n, b_\lambda^n, \underline{a}_\lambda^n, \underline{\tilde{a}}_\lambda^n, \underline{b}_\lambda^n$ are arbitrary constants

E. The Composite Expansion (Appendix B)

$\mathbf{q}_i^* = u_i^* \mathbf{i}_x + v_i^* \mathbf{i}_\rho + w_i^* \mathbf{i}_\theta$ = inner approximation to $\mathbf{q}^*$

$\mathbf{q}_o^* = u_o^* \mathbf{i}_x + v_o^* \mathbf{i}_\rho + w_o^* \mathbf{i}_\theta$ = outer approximation to $\mathbf{q}^*$

$\mathbf{q}_c^* = u_c^* \mathbf{i}_x + v_c^* \mathbf{i}_\rho + w_c^* \mathbf{i}_\theta$ = common part of $\mathbf{q}_i^*$ and $\mathbf{q}_o^*$

p_i^* = inner approximation to p^*

p_o^* = outer approximation to p^*

p_c^* = common part of p_i^* and p_o^*

NOMENCLATURE (Cont'd)

F. Mathematical Symbols

$\mathbf{a} \circ \mathbf{b}$ = dyadic product of vectors $\mathbf{a}$, $\mathbf{b}$; $(\mathbf{a} \circ \mathbf{b}) \cdot \mathbf{c} = \mathbf{a}(\mathbf{b} \cdot \mathbf{c})$ for any vector $\mathbf{c}$

$\underline{\underline{A}} : \underline{\underline{B}}$ = scalar product of tensors $\underline{\underline{A}}$ and $\underline{\underline{B}}$;

$$\underline{\underline{A}} : \underline{\underline{B}} = \sum_{i,j} A_{ij} B_{ji}; \quad A_{ij},\ B_{ij} = \text{cartesian components}$$

$$H_n(W) = \frac{\partial W}{\partial \bar{x}} - \frac{1}{\mathfrak{Re}} \left(\frac{\partial^2 W}{\partial \bar{\rho}^2} + \frac{1}{\bar{\rho}} \frac{\partial W}{\partial \bar{\rho}} - \frac{n^2}{\bar{\rho}^2} W \right) = \frac{\partial W}{\partial \bar{x}} - \frac{1}{\mathfrak{Re}} L_n(W)$$

f_ν, g_ν, h_ν = forcing terms

$$\mathrm{Ei}(-z) = -\int_z^\infty \frac{e^{-t}}{t}\, dt = \gamma + \log z + \sum_{n=1}^{\infty} \frac{(-z)^n}{n!\, n} \sim -\frac{e^{-z}}{z} \quad \text{as } z \to \infty,\ z > 0.$$

γ = Euler's constant = 0.5772+

REFERENCES

1. Imai, I., "On the Asymptotic Behavior of Viscous Fluid Flow at a Great Distance from a Cylindrical Body, with Special Reference to Filon's Paradox," *Proceedings of the Royal Society of London, Series A*, Vol. 208, 1951, pp. 487-516.

2. Chang, I-Dee, "Navier-Stokes Solutions at Large Distances from a Finite Body," *Journal of Mathematics and Mechanics*, Vol. 10, 1961, pp. 811-876.

3. Lagerstrom, P. A., and Cole, J. D., "Examples Illustrating Expansion Procedures for the Navier-Stokes Equations," *Journal of Rational Mechanics and Analysis*, Vol. 4, 1955, pp. 817-882.

4. Kaplun, S., and Lagerstrom, P. A., "Asymptotic Expansions of Navier-Stokes Solutions for Small Reynolds Numbers," *Journal of Mathematics and Mechanics*, Vol. 6, 1957, pp. 585-593.

5. *Higher Transcendental Functions*, Vol. 1, Bateman Manuscript Project, California Institute of Technology, McGraw-Hill Book Co., Inc., New York, 1953.

6. Copson, E. T., *Introduction to the Theory of Functions of a Complex Variable*, Oxford University Press, Inc., New York, 1935.

7. Garstang, T. E., "The Forces on a Solid Body in a Stream of Viscous Fluid," *Philosophical Transactions of the Royal Society of London*, 1936, pp. 25-75.

APPENDIX A

I. THE NAVIER-STOKES EQUATIONS

In dimensionless cylindrical polar coordinates (see Section III-A), the Navier-Stokes equations for the stationary flow of a viscous, incompressible fluid are

$$u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial \rho^*} + \frac{w^*}{\rho^*} \frac{\partial p^*}{\partial \theta} + \frac{\partial p^*}{\partial x^*} - \frac{1}{\Re_e} \nabla^{*2} u^* = 0 \tag{A-1a}$$

$$u^* \frac{\partial v^*}{\partial x^*} + v^* \frac{\partial v^*}{\partial \rho^*} + \frac{w^*}{\rho^*} \frac{\partial v^*}{\partial \theta} - \frac{w^{*2}}{\rho^*} + \frac{\partial p^*}{\partial \rho^*}$$
$$- \frac{1}{\Re_e} \left(\nabla^{*2} v^* - \frac{v^*}{\rho^{*2}} - \frac{2}{\rho^{*2}} \frac{\partial w^*}{\partial \theta} \right) = 0 \tag{A-1b}$$

$$u^* \frac{\partial w^*}{\partial x^*} + v^* \frac{\partial w^*}{\partial \rho^*} + \frac{w^*}{\rho^*} \frac{\partial w^*}{\partial \theta} + \frac{v^* w^*}{\rho^*} + \frac{1}{\rho^*} \frac{\partial p^*}{\partial \theta}$$
$$- \frac{1}{\Re_e} \left(\nabla^{*2} w^* - \frac{w^*}{\rho^{*2}} + \frac{2}{\rho^{*2}} \frac{\partial v^*}{\partial \theta} \right) = 0 \tag{A-1c}$$

$$\frac{\partial (\rho^* u^*)}{\partial x^*} + \frac{\partial (\rho^* v^*)}{\partial \rho^*} + \frac{\partial w^*}{\partial \theta} = 0 \tag{A-1d}$$

where

$$\nabla^{*2} = \frac{\partial^2}{\partial x^{*2}} + \frac{\partial^2}{\partial \rho^{*2}} + \frac{1}{\rho^*} \frac{\partial}{\partial \rho^*} + \frac{1}{\rho^{*2}} \frac{\partial^2}{\partial \theta^2} \tag{A-1e}$$

The divergence of the momentum equation becomes

$$\nabla^{*2} p^* + 2\left(\frac{\partial u^*}{\partial x^*}\right)^2 + 2\left(\frac{\partial v^*}{\partial \rho^*}\right)^2 + 2\,\frac{\partial u^*}{\partial x^*}\,\frac{\partial v^*}{\partial \rho^*} + 2\,\frac{\partial v^*}{\partial x^*}\,\frac{\partial u^*}{\partial \rho^*}$$

$$+ \frac{2}{\rho^*}\,\frac{\partial w^*}{\partial x^*}\,\frac{\partial u^*}{\partial \theta} + \frac{2}{\rho^*}\,\frac{\partial w^*}{\partial \rho^*}\,\frac{\partial v^*}{\partial \theta} - \frac{2}{\rho^*}\,w^*\,\frac{\partial w^*}{\partial \rho^*} = 0 \qquad \text{(A-2)}$$

II. SIMILARITY SOLUTIONS OF THE EQUATION $H_n(W) = 0$

We seek solutions of the homogeneous equation $H_n(W) = 0$ which are of the form

$$W = \frac{\sigma^n}{\bar{\rho}^n \bar{x}^{m-n/2}}\,\psi(\sigma) \qquad \text{(A-3)}$$

where $n = 0, 1, 2, \cdots, m \geq 0$, and σ is defined by

$$\sigma = \frac{\mathcal{R}_e}{4}\,\frac{\bar{\rho}^2}{\bar{x}} = \frac{\mathcal{R}_e}{4}\,\frac{\rho^{*2}}{x^*} \qquad \text{(A-4)}$$

By direct substitution,

$$H_n(W) = -\frac{\sigma^n}{\bar{\rho}^n \bar{x}^{m-n/2+1}}\,D_m^n(\psi) \qquad \text{(A-5)}$$

where

$$D_m^n(\psi) = \sigma\psi'' + (\sigma + n + 1)\psi' + (m + n/2)\psi \qquad \text{(A-6)}$$

the prime denoting differentiation with respect to σ. Two linearly independent solutions of the equation $D_m^n(\psi) = 0$ are

$$\psi_1 = \Phi(m + n/2,\ n + 1;\ -\sigma) \tag{A-7a}$$

$$\psi_2 = \psi_1 \int^{\sigma} e^{-s}\, s^{-(n+1)} [\psi_1(s)]^{-2}\, ds \tag{A-7b}$$

where Φ denotes the confluent hypergeometric function. Since $\Phi(a, b; 0) = 1$, one sees from Eq. (A-7b) that ψ_2 has a pole of order $n\,(n = 1, 2, \cdots)$ at the origin and behaves as $\log \sigma$ if $n = 0$. Thus for all non-negative integers n, $\bar{W}$ is regular on the positive $\bar{x}$-axis if and only if ψ is a multiple of ψ_1.

As $\sigma \to \infty$, we have (cf. Ref. 6, p. 265)

$$\psi_1 \sim \frac{\Gamma(n + 1)}{\Gamma(1 - m + n/2)}\, \sigma^{-(m+n/2)}\ {}_2F_0\,(m + n/2,\ m - n/2;\ \sigma^{-1}) \tag{A-8}$$

One sees from this expansion that ψ_1 decays exponentially if and only if

$$m - n/2 = 1, 2, \cdots \tag{A-9}$$

If $2m = n$, Eq. (A-8) may be replaced by the expansion

$$\psi_1 \sim \frac{n!}{\sigma^n} + \cdots \tag{A-10}$$

where the dots indicate a transcendentally small remainder. The right-hand side of Eq. (A-10) is replaced by $\log \sigma$ for the case $n = 0$. Comparing Eq. (A-3) and (A-10), one sees that there exist solutions of $H_n(\bar{W}) = 0$ whose algebraically decaying part is a function of $\bar{\rho}$ alone; for these solutions $2m = n = 0, 1, 2, \cdots$. These exceptional solutions appear as eigensolutions in the inner expansion of the transverse velocity components. The case $m = n = 0$ is then eliminated by matching.

The solutions $\bar{W}$ of interest in Parts III and IV are regular on the positive $\bar{x}$-axis and satisfy homogeneous matching conditions at $\bar{\rho} = \infty$. We define the eigensolutions

$$\bar{W}_m^n = \frac{\sigma^n}{\bar{\rho}^n \bar{x}^{m-n/2}} \Phi(m + n/2, m + 1; -\sigma) \tag{A-11}$$

where $m - n/2 = 1, 2, \cdots, n = 0, 1, \cdots$, or $2m = n = 1, 2, \cdots$. The desired eigensolutions are constant multiples of the solutions $\bar{W}_m^n$.

III. SEVERAL SOLUTIONS OF LAPLACE'S EQUATION

We consider first solutions of Laplace's equation which are of the form

$$\begin{aligned} R_\lambda^n(\tilde{r}, \mu, \theta) &= B_\lambda^n(\tilde{r}, \mu) (\sin n\theta, \cos n\theta) \\ &= \tilde{r}^{-(\lambda+1)} G_\lambda^n(\mu) (\sin n\theta, \cos n\theta) \qquad n = 0, 1, 2, \cdots \end{aligned} \tag{A-12}$$

where

$$\tilde{r} = (\tilde{x}^2 + \tilde{\rho}^2)^{1/2}, \qquad \mu = \frac{\tilde{x}}{\tilde{r}} = \frac{x^*}{r^*} \tag{A-13}$$

By direct substitution, G_λ^n is a solution of Legendre's differential equation. If G_λ^n is required to be a regular function of μ on the interval $-1 \leq \mu \leq +1$, then $\lambda = 0, 1, 2, \cdots$ and $G_\lambda^n = P_\lambda^n$, the associated Legendre functions of the first kind. Under these conditions R_λ^n is, within a multiplicative constant, the potential of a term in the outer expansion of $-\mathbf{q}^* \cdot i$ which matches with the eigensolutions of order $\nu = \lambda + n/2 + 3/2$ in the inner expansion of $\bar{p}$. Since $P_\lambda^n = 0$ whenever $n - \lambda = 1, 2, \cdots$, we have, by matching,

$$\tilde{a}_\nu^n = \tilde{\underline{a}}_{-\nu}^n = 0, \qquad (3/2n - \nu + 1/2 = 0, 1, 2, \cdots) \tag{A-14}$$

in the eigensolution Eq. (65).

In order to match the inner and outer expansions of v^* and w^* for the general three-dimensional case, additional solutions of the form (A-12) are required for which $\lambda = n - 1$. For details concerning the solutions of Legendre's equation applicable to these cases, see Ref. 7. For our purposes, we define

$$G^n_{n-1}(\mu) = \frac{\Gamma(2n-1)}{2^{2n-1}[\Gamma(n)]^2} \frac{(1+\mu)^{n/2}}{(1-\mu)^{1-n/2}} F\left(1,\ 1-n,\ 2-2n;\ \frac{2}{1-\mu}\right) \tag{A-15}$$

where F denotes the hypergeometric function. The function $-cR^n_{n-1}$ (c = arbitrary constant) is the potential of a term in the outer expansion of $\mathbf{q}^*$ which matches with the eigensolutions $cW^n_{n/2}$ $(\sin n\theta, \cos n\theta)$ appearing in the inner expansion of v^* and w^*.

A final class of harmonic functions which are required in the construction have the form

$$S^n_\lambda(\tilde{r}, \mu, \theta) = C^n_\lambda(\tilde{r}, \mu)\ (\sin n\theta, \cos n\theta)$$

$$= \tilde{r}^{-(\lambda+1)} [K^n_\lambda(\mu) - G^n_\lambda(\mu) \cdot \log \tilde{r}]\ (\sin n\theta, \cos n\theta) \tag{A-16}$$

The first few such solutions are defined by

$$G^0_1 = \mu \tag{A-17a}$$

$$K^0_1 = \mu \log(1-\mu) + 1 \tag{A-17b}$$

$$G^1_1 = -(1-\mu^2)^{1/2} \tag{A-17c}$$

$$K^1_1 = \left(\frac{1+\mu}{1-\mu}\right)^{1/2} - (1-\mu^2)\log(1-\mu) \tag{A-17d}$$

$$G^2_1 = -\frac{1}{4}\left(\frac{\mu^2-\mu-2}{1-\mu}\right) \tag{A-17e}$$

$$K^2_1 = \frac{1}{4}\left[1 + 2\mu + \left(\frac{\mu^2+\mu-2}{1+\mu}\right)\log\left(\frac{1-\mu}{2}\right)\right] \tag{A-17f}$$

The S^n_λ are regular everywhere except at the origin and along the positive $\tilde{x}$-axis. Note also that these solutions always require switchback terms (see Section III-F). The construction of these solutions follows in a straightforward way from the assumed form and is omitted.

APPENDIX B

I. THE COMPOSITE EXPANSION FOR THE AXIALLY SYMMETRIC CASE

$$q^* = \mathbf{q}_o^* + \mathbf{q}_i^* - \mathbf{q}_c^* + o(r^{*-2})\,\mathbf{i}_x + o(r^{*-5/2})\,\mathbf{i}_\rho + o(r^{*-5/2})\,\mathbf{i}_\theta$$

$$p^* = p_o^* + p_i^* - p_c^* + o(r^{*-3})$$

where

$$\mathbf{q}_o^* = \mathbf{i} + \nabla^* \left\{ -\frac{a}{4\pi}\frac{1}{r^*} - \frac{a^2}{2\mathcal{R}_e}\frac{1}{r^{*3}}\left[x^* \log(r^* - x^*) + r^* - 2x^* \log r^*\right] + \frac{\tilde{a}_1}{2}\frac{x^*}{r^{*3}} \right\}$$

$$p_o^* = 1 - u_o^*$$

$$u_i^* = 1 - \alpha\frac{e^{-\sigma}}{x^*} - \frac{a}{4\pi}\frac{1}{x^{*2}}(\sigma^2 - 4\sigma + 1)\,e^{-\sigma}$$

$$+ \frac{\alpha^2}{4}\frac{1}{x^{*2}}\left\{(1-\sigma)\,e^{-\sigma}\left[\mathrm{Ei}(-\sigma) - \log\sigma + \log x^*\right] - 2e^{-\sigma} - e^{-2\sigma}\right\} + \frac{a_1}{x^{*2}}(1-\sigma)\,e^{-\sigma}$$

$$v_i^* = -\frac{a}{2\pi}\frac{\sigma e^{-\sigma}}{\rho^* x^*} + \frac{a}{2\pi\mathcal{R}_e}\frac{\sigma}{\rho^* x^{*2}}(\sigma^2 - 2\sigma)\,e^{-\sigma}$$

$$+ \frac{\alpha^2}{2\mathcal{R}_e}\frac{\sigma}{\rho^* x^{*2}}\left\{(2-\sigma)\,e^{-\sigma}\left[\mathrm{Ei}(-\sigma) - \log\sigma + \log x^*\right]\right.$$

$$\left. + \frac{1}{\sigma}\left(e^{-\sigma} + e^{-2\sigma} - 2\right) - 3e^{-\sigma} - e^{-2\sigma}\right\} + \frac{2a_1}{\mathcal{R}_e}\frac{\sigma}{\rho^* x^{*2}}(2-\sigma)\,e^{-\sigma}$$

$$w_i^* = -\beta \frac{\sigma}{\rho^* x^*} e^{-\sigma} - \frac{\beta\alpha}{8\rho^* x^{*2}} \sigma \left\{ \frac{1}{\sigma} (e^{-2\sigma} - e^{-\sigma}) + e^{-2\sigma} + 3 e^{-\sigma} \right.$$

$$\left. + (\sigma - 2) e^{-\sigma} [\mathrm{Ei}(-\sigma) - \log \sigma + \log x^*] \right\} + \frac{\beta}{\Re_e} \frac{\sigma}{\rho^* x^{*2}} (\sigma^2 - 2\sigma - 2) e^{-\sigma}$$

$$+ m_1 \frac{\sigma}{\rho^* x^{*2}} (2 - \sigma) e^{-\sigma}$$

$$p_i^* = -\frac{a}{4\pi} \frac{1}{x^{*2}} + \frac{3a}{8\pi} \frac{\rho^{*2}}{x^{*4}} + \frac{\alpha^2}{2\Re_e} \frac{1}{x^{*3}} \left[2\,\mathrm{Ei}(-2\sigma) - \frac{1}{2} e^{-2\sigma} - 2 \log \sigma \right]$$

$$- \frac{\Re_e}{16} \beta^2 \frac{e^{-2\sigma}}{x^{*3}} + \frac{2\alpha^2}{\Re_e} \frac{\log x^*}{x^{*3}} + \left[\frac{\alpha^2}{2\Re_e} (2\Re_e - 2 \log 2 - 5) + \tilde{a}_1 \right] \frac{1}{x^{*3}}$$

$$u_c^* = 1 + \frac{a}{4\pi} \frac{1}{x^{*2}}$$

$$v_c^* = \frac{a}{4\pi} \frac{\rho^*}{x^{*3}} - \frac{\alpha^2}{\Re_e} \frac{1}{\rho^* x^{*2}}$$

$$w_c^* = 0$$

$$p_c^* = -\frac{a}{4\pi} \frac{1}{x^{*2}} + \frac{3a}{4\pi} \frac{\rho^{*2}}{x^{*4}} - \frac{2\alpha^2}{\Re_e} \frac{\log \rho^*}{x^{*3}} + \frac{3\alpha^2}{\Re_e} \frac{\log x^*}{x^{*3}} + \frac{\alpha^2}{2\Re_e} (2 \log 2 - 5) \frac{1}{x^{*3}} + \frac{\tilde{a}_1}{x^{*3}}$$

Here

$$\alpha = \frac{a\Re_e}{4\pi}, \qquad \beta = \frac{m\Re_e}{4\pi}$$

$$\sigma = \frac{\Re_e \rho^{*2}}{4x^*}$$

II. THE COMPOSITE EXPANSION FOR THE GENERAL THREE-DIMENSIONAL CASE

$$\mathbf{q}^* = \mathbf{q}_o^* + \mathbf{q}_i^* - \mathbf{q}_c^* + o(r^{*-3/2})$$

$$p^* = p_o^* + p_i^* - p_c^* + o(r^{*-2})$$

where

$$\mathbf{q}_o^* = \mathbf{i} + \nabla^* \left\{ -\frac{a}{4\pi}\frac{1}{r^*} + \frac{b}{4\pi}\frac{\rho^* \sin\theta}{(r^* - x^*)r^*} \right\}$$

$$p_o^* = 1 - u_o^*$$

$$u_i^* = -\frac{\alpha}{\bar{x}} e^{-\sigma} + \frac{1}{2}\alpha\omega\frac{\sigma\sin\theta}{\rho^* x^*}\left[e^{-\sigma}\,\mathrm{Ei}(-\sigma) + \frac{1}{\sigma}(e^{-2\sigma} - e^{-\sigma}) - e^{-\sigma}\log\sigma\right.$$

$$\left. + (\log x^* - \gamma + 1)e^{-\sigma}\right] + \frac{\sigma e^{-\sigma}}{\rho^* x^*}(a_1^1 \sin\theta + \underline{a}_1^1 \cos\theta)$$

$$v_i^* = \frac{b}{2\pi}\frac{(e^{-\sigma} - 1)}{\rho^{*2}}\sin\theta - \frac{a}{2\pi}\frac{\sigma e^{-\sigma}}{\rho^* x^*} + \frac{2\omega^2}{\Re_e}\frac{\cos 2\theta}{\rho^{*3}}G(\sigma; x^*)$$

$$+ \frac{2}{\rho^{*3}}(1 - e^{-\sigma} - \sigma e^{-\sigma})(b_{3/2}^2 \sin 2\theta + \underline{b}_{3/2}^2 \cos\theta)$$

$$w_i^* = -\frac{b}{2\pi}\frac{1}{\rho^{*2}}\left[(2\sigma+1)e^{-\sigma}-1\right]\cos\theta - \beta\frac{\sigma e^{-\sigma}}{\rho^* x^*}$$

$$+\frac{2\omega^2}{\Re_e}\frac{\sin 2\theta}{\rho^{*3}}\left[G(\sigma;x^*) - \sigma\frac{\partial}{\partial\sigma}G(\sigma;x^*)\right]$$

$$+\frac{2}{\rho^{*3}}\left(1-e^{-\sigma}-\sigma e^{-\sigma}-\sigma^2 e^{-\sigma}\right)\left(\underline{b}_{3/2}^2\sin\theta - b_{3/2}^2\cos\theta\right)$$

$$p_i^* = -\frac{a}{4\pi}\frac{1}{x^{*2}} + \frac{\omega^2}{4x^{*2}}\left[\mathrm{Ei}(-\sigma) - \mathrm{Ei}(-2\sigma) + \frac{1}{\sigma}\left(e^{-\sigma}-e^{-2\sigma}\right) + \frac{1}{2\sigma^2}\left(2e^{-\sigma}-e^{-2\sigma}-1\right)\right]$$

$$+\frac{\omega^2}{4}\frac{\cos 2\theta}{x^{*2}}\left[\sigma\mathrm{Ei}(-2\sigma) - \sigma\mathrm{Ei}(-\sigma) + \frac{1}{2}\left(e^{-2\sigma}-2e^{-\sigma}\right) - \frac{1}{4\sigma}\left(3e^{-2\sigma}-4e^{-\sigma}\right) - \frac{1}{4\sigma}\right]$$

$$u_c^* = 0$$

$$v_c^* = -\frac{b\sin\theta}{2\pi\rho^{*2}} - \frac{\omega^2}{2\Re_e}\frac{\log x^*}{\rho^{*3}}\cos 2\theta + \frac{2}{\rho^{*3}}\left(b_{3/2}^2\sin 2\theta + \underline{b}_{3/2}^2\cos 2\theta\right)$$

$$w_c^* = \frac{b}{2\pi}\frac{\cos\theta}{\rho^{*2}} - \frac{\omega^2}{2\Re_e}\frac{\log x^*}{\rho^{*3}}\sin 2\theta + \frac{2}{\rho^{*3}}\left(\underline{b}_{3/2}^2\sin 2\theta - b_{3/2}^2\cos 2\theta\right)$$

$$p_c^* = -\frac{a}{4\pi}\frac{1}{x^{*2}}$$

Here

$$\omega = \frac{b\Re_e}{4\pi}$$

$$G(\sigma;x^*) = \frac{\sigma^2}{2}\,[\mathrm{Ei}(-2\sigma) - \mathrm{Ei}(-\sigma)] + \frac{\sigma}{4}\,[e^{-2\sigma} - 2e^{-\sigma} - e^{-\sigma}\log\sigma + e^{-\sigma}\,\mathrm{Ei}(-\sigma)]$$

$$+ \frac{1}{8}\,[e^{-2\sigma} - 4e^{-\sigma} + 2e^{-\sigma}\,\mathrm{Ei}(-\sigma) - 2e^{-\sigma}\log\sigma - 2\,\mathrm{Ei}(-2\sigma) + 2\,\mathrm{Ei}(-\sigma)]$$

$$+ \frac{1}{4}\,(3/2 + \log 2 - \gamma)(\sigma + 1)\,e^{-\sigma} + \frac{1}{4}\,(\log x^*)(\sigma + 1)\,e^{-\sigma} - \frac{1}{4}\log x^*$$